computing connection matrices

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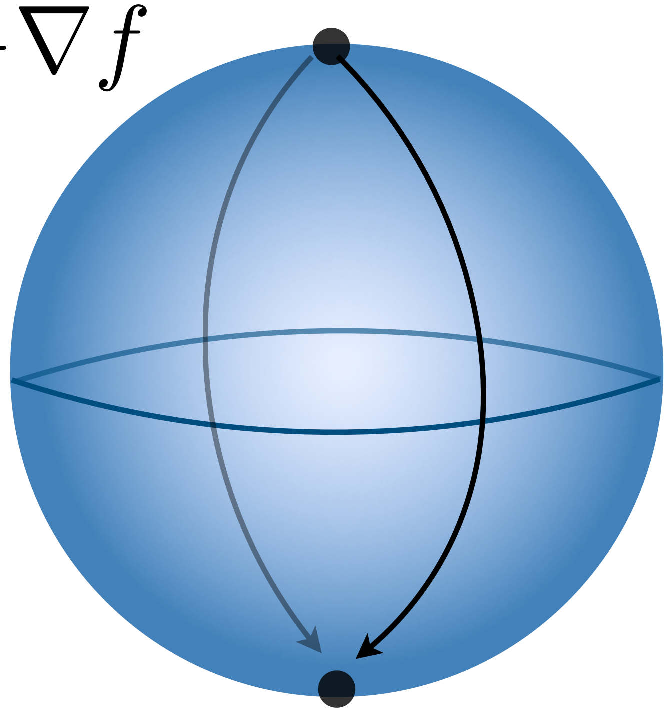
philosophy

- a dynamical system assembles topological data
- the data are ordered and measured with algebra
- continuous $\xleftrightarrow{\textit{computational Conley theory}}$ combinatorial
- connection matrix is a representative of equivalence class

$$\dot{x} = -\nabla f$$



$\xrightarrow{f}$

order theory

data are *ordered* by dynamics

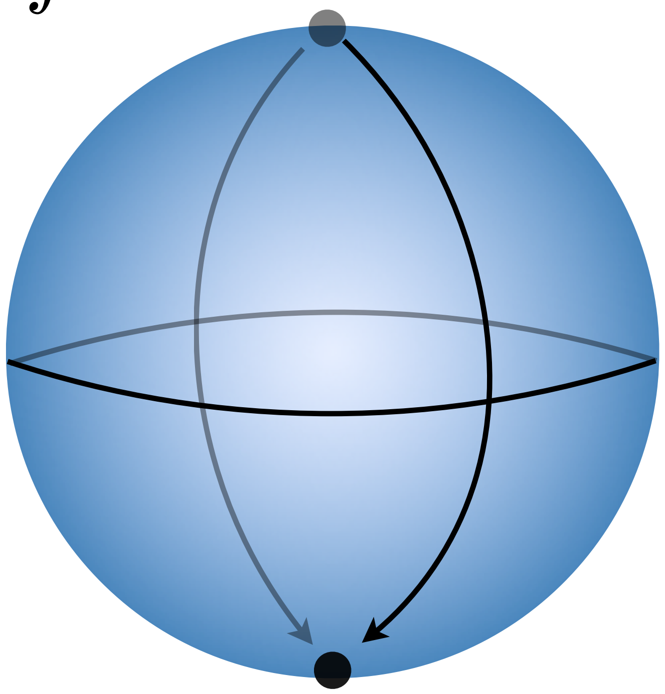
algebraic topology

dynamics *measure* data via algebraic invariants

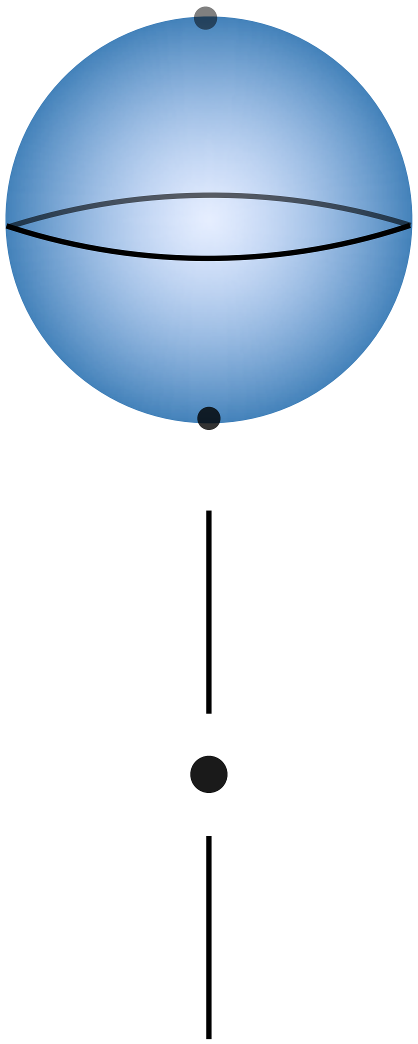
lattice of attractors

attractors are topological data

$$\dot{x} = -\nabla f$$



attractors are *ordered* by dynamics

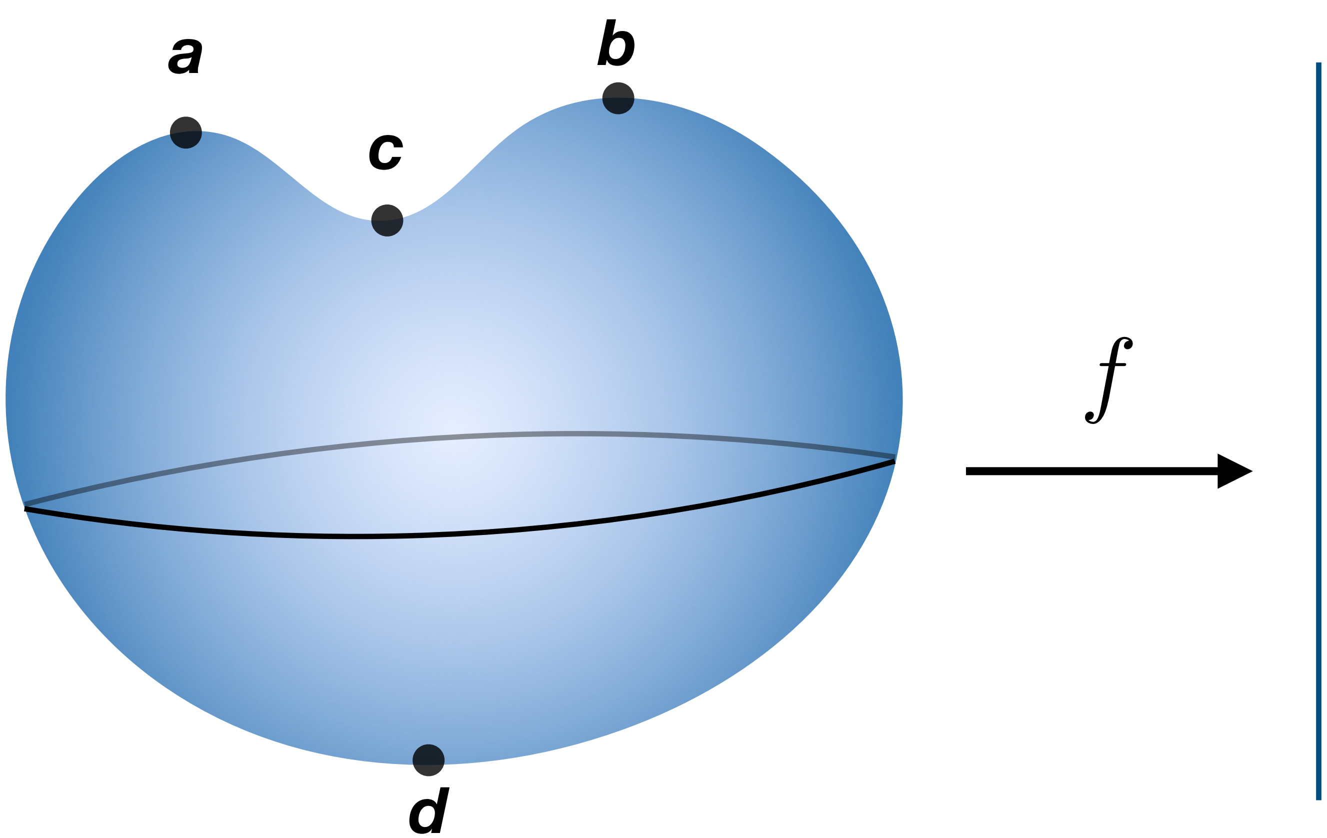


A is an attractor if $\exists$ (regular,closed) $N \supseteq A$ such that $\omega(N) = A$

for a critical $f^{-1}(-\infty, a]$ is an attractor

Morse theory

equilibria are topological data



equilibria are *measured* via algebraic invariants

Morse complex
$M_\bullet$

$$\begin{array}{c} 0 \\ \downarrow \\ \mathbb{Z}_2\langle a\rangle \oplus \mathbb{Z}_2\langle b\rangle \\ \Big\downarrow {\scriptstyle \begin{pmatrix} 1 & 1 \end{pmatrix}} \\ \mathbb{Z}_2\langle c\rangle \\ \Big\downarrow {\scriptstyle 0} \\ \mathbb{Z}_2\langle d\rangle \\ \downarrow \\ 0 \end{array}$$

Morse theory is the prototypical example

$$H(M_\bullet(f)) \cong H(X)$$

Morse index is a measurement of instability

lattice of subcomplexes

a *subcomplex* is a collection of subspaces where the inclusions form chain maps

$B_n \subseteq C_n$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r] \arrow[u, "i_{n+1}"] & B_n \arrow[r] \arrow[u, "i_n"] & B_{n-1} \arrow[r] \arrow[u, "i_{n-1}"] & \cdots
\end{tikzcd}$$

lattice of subcomplexes

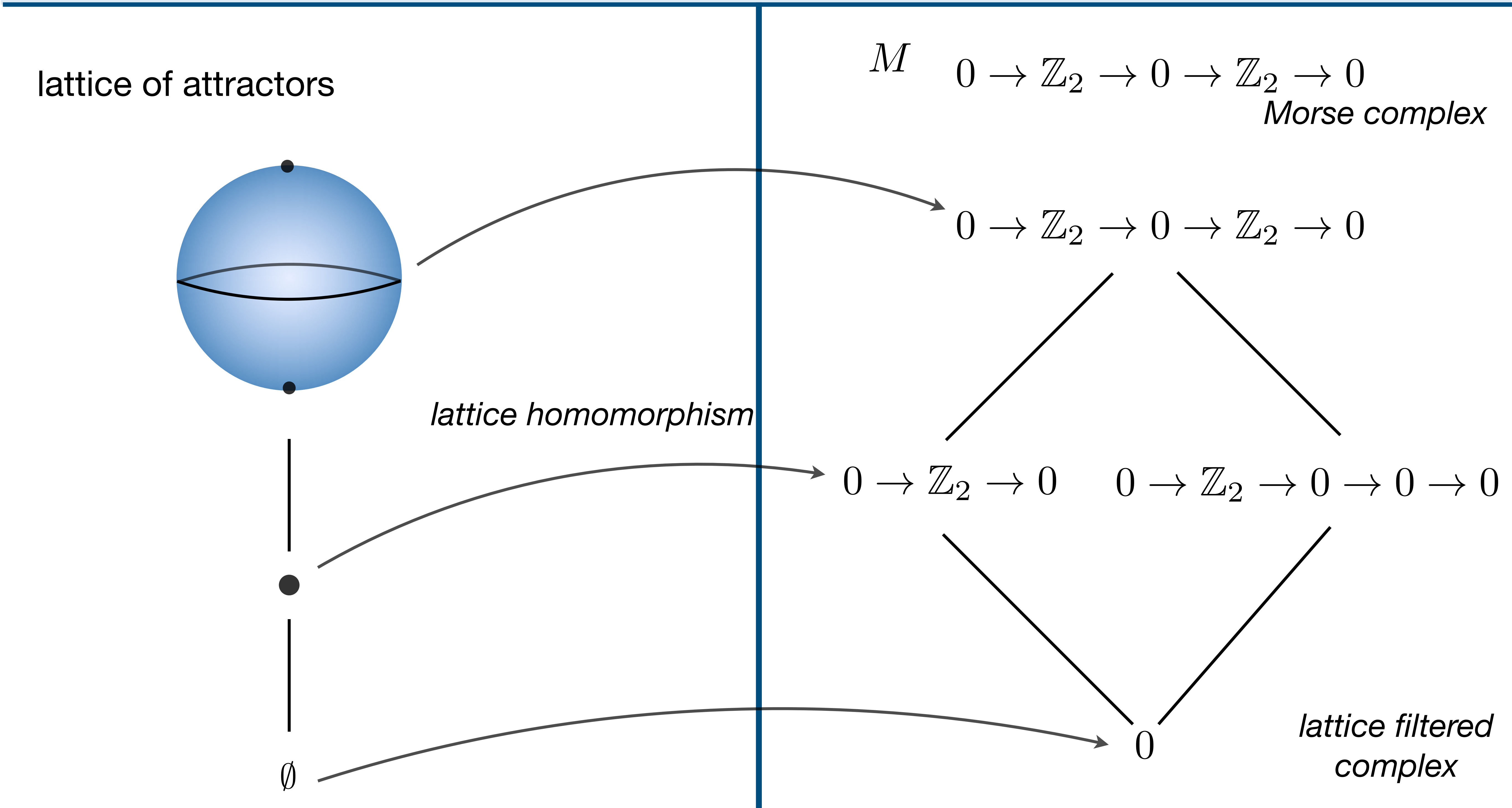
$\cap \quad +$

$$\begin{tikzcd}
 & 0 \to \mathbb{Z}_2 \to 0 \to \mathbb{Z}_2 \to 0 \arrow[dl, dash] \arrow[dr, dash] & \\
0 \to 0 \to 0 \to \mathbb{Z}_2 \to 0 \arrow[dr, dash] & & 0 \to \mathbb{Z}_2 \to 0 \to 0 \to 0 \arrow[dl, dash] \\
 & 0 &
\end{tikzcd}$$

join irreducible elements (poset)

$$J(L) := \{x \in L : \text{ if } x = a + b \text{ then } x = a \text{ or } x = b\}$$

a Morse representation

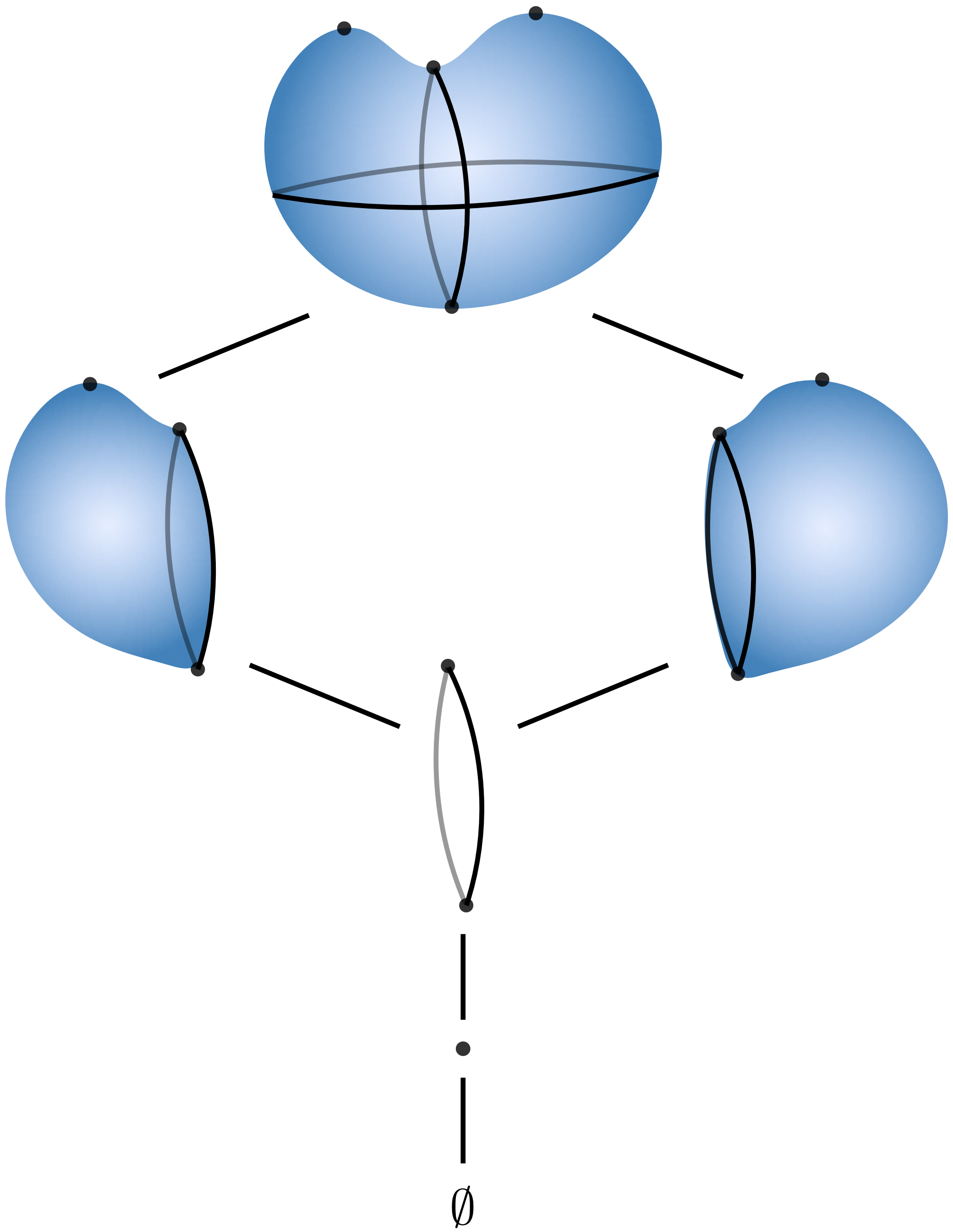


a bridge between order theory and algebraic topology

$$\text{for } a \text{ critical } f^{-1}(-\infty, a] \rightsquigarrow \{\mathbb{Z}_2\langle b\rangle : f(b) \leq f(a)\}$$

a Conley representation

lattice of attractors



Morse complex

$$0 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to 0$$

$$0 \to \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to 0$$

$$0 \to \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to 0$$

$$0 \to \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to 0$$

$$0 \to \mathbb{Z}_2 \to 0$$

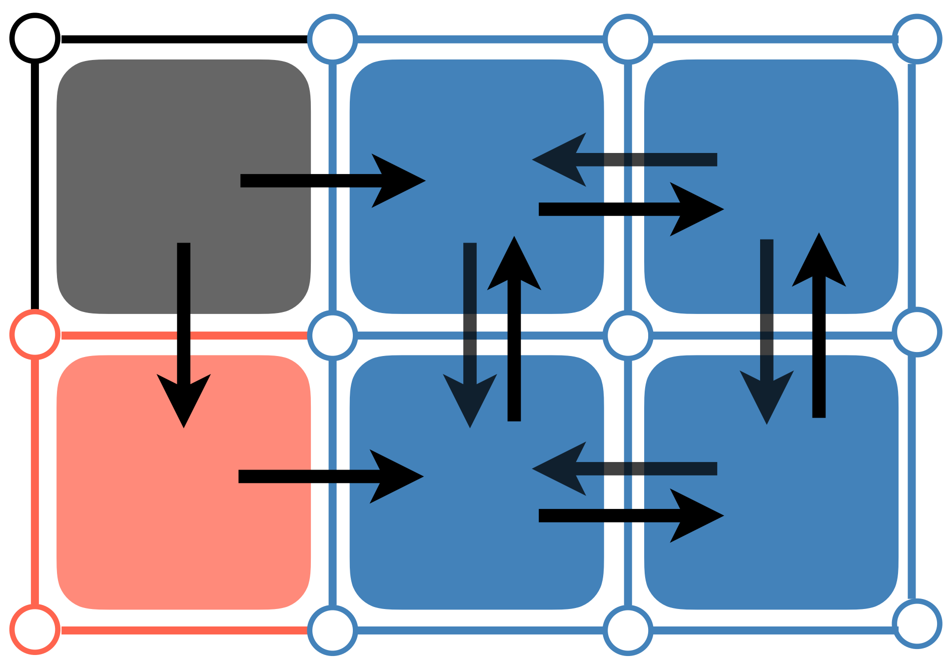
$$0$$

Conley's insight: attractors organize the global behavior

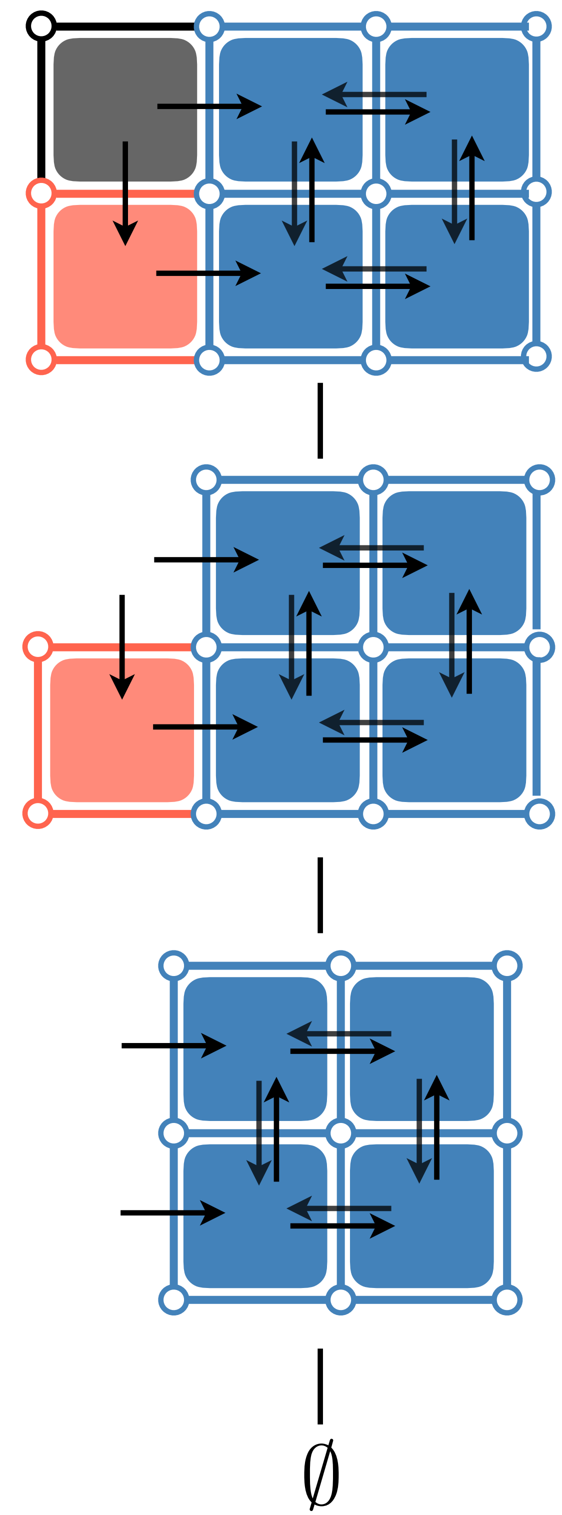
…filter by attractors (not sublevel sets)

as a data structure

dynamics on X…



…gives a lattice of attractors



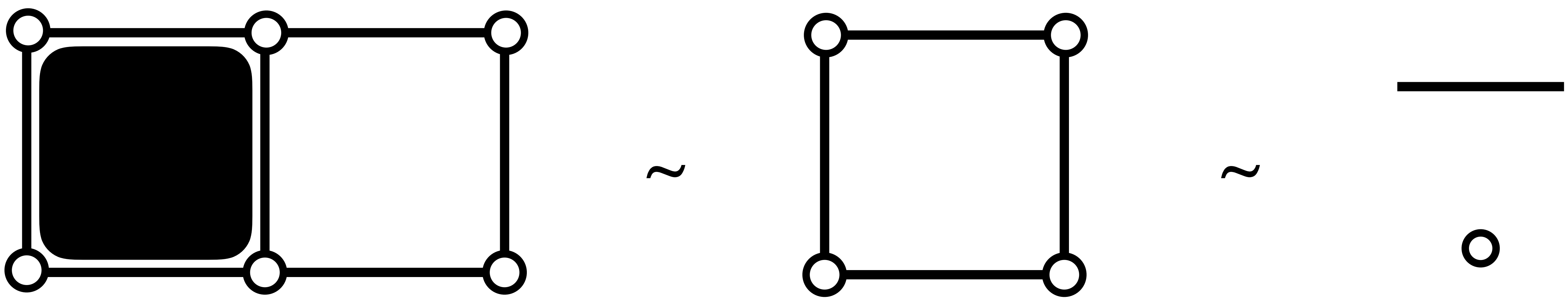
$p \in J(L) \qquad X_p / X_{Pred(p)}$

Conley index $\quad H(X_p / X_{Pred(p)})$

we care about the algebraic invariants of this structure: homology, relative homology

homology as data reduction

often we only care about chain complexes up to homology…



…so it is natural to make an equivalence class

$$B \sim_D C \text{ if there is a chain map } B \xrightarrow{\phi} \text{ with } H(B) \overset{\phi*}{\cong} H(C)$$
$$B \sim_K C \text{ if } B, C \text{ are chain homotopy equivalent}$$

fields

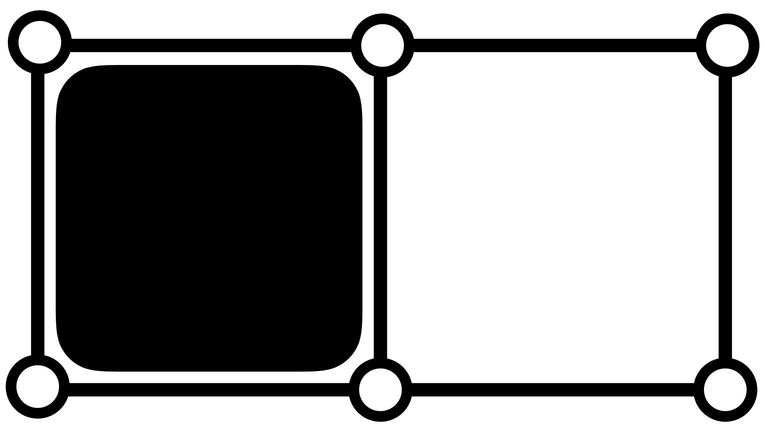
homology is a (simple, minimal) representative of this equivalence class

$$\begin{tikzcd}
\dots \arrow[r, "d"] & C_{n+1} \arrow[r, "d"] \arrow[d, "\phi_{n+1}"] & C_n \arrow[r, "d"] \arrow[d, "\phi_n"] & C_{n-1} \arrow[r, "d"] \arrow[d, "\phi_{n-1}"] & \dots \\
\dots \arrow[r, "0"] & H_{n+1} \arrow[r, "0"] & H_n \arrow[r, "0"] & H_{n-1} \arrow[r, "0"] & \dots
\end{tikzcd}$$

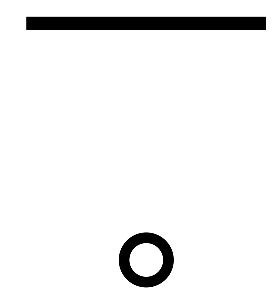
zero differentials

discrete Morse theory

regular CW complex



discrete Morse theory



$$(C(X), \partial) \longrightarrow (H(C(X)), 0)$$

black box for computing homology

homology as simple representative

equivalent filtered complex

replace with equivalent representation

homotopy equivalence

$$\begin{tikzcd}
 & M \arrow[dl, dash] \arrow[dr, dash] \arrow[rrr, "\phi"] & & & A \arrow[dl, dash] \arrow[dr, dash] & \\
M_2 \arrow[dr, dash] \arrow[rrr, bend left] & & M_3 \arrow[dl, dash] \arrow[rrr, bend left] & A_2 \arrow[dr, dash] & & A_3 \arrow[dl, dash] \\
 & M_4 \arrow[d, dash] \arrow[rrr, bend left] & & & A_4 \arrow[d, dash] & \\
 & 0 \arrow[rrr] & & & 0 &
\end{tikzcd}$$

$\phi(M_i) \subseteq A_i$

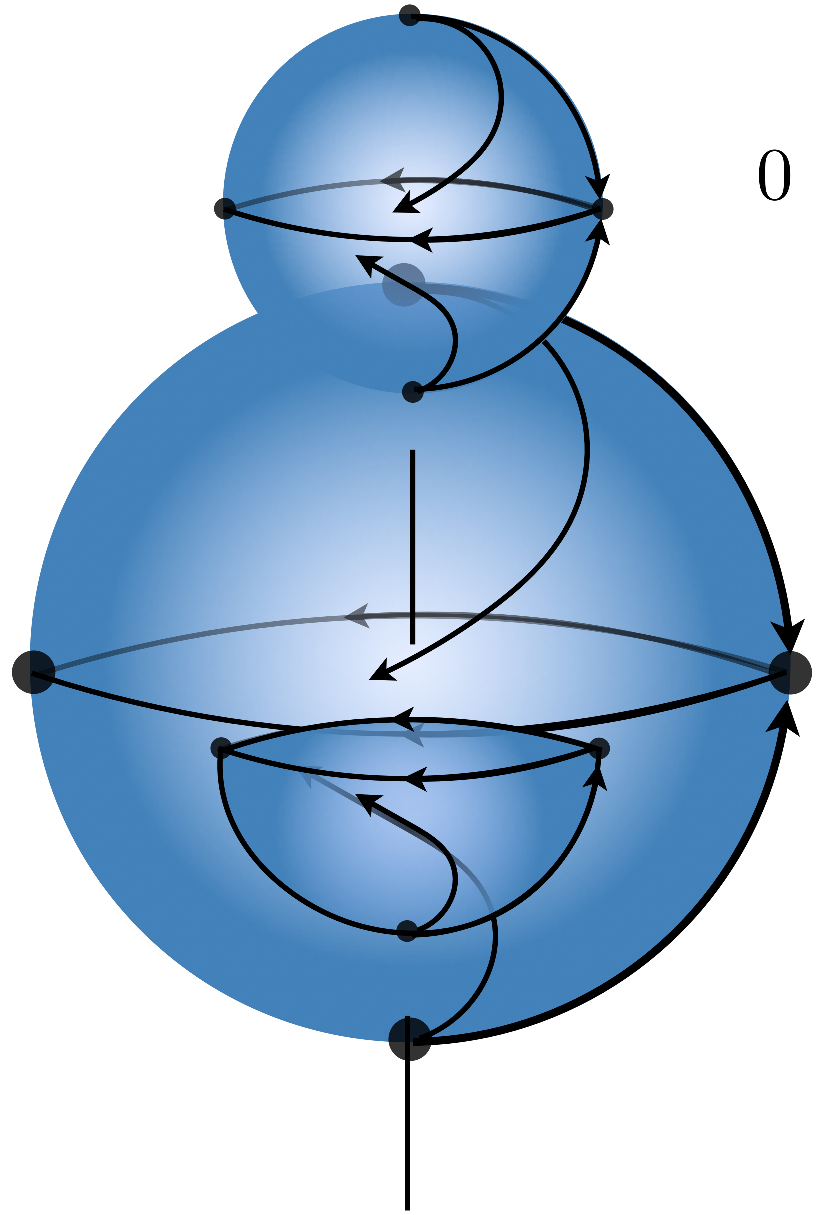
filtered homotopy equivalence

$(M, L) \sim (A, L)$

consequence: $H(M_i) \cong H(A_i)$

homotopy category for lattice filtered complexes

Conley data



$\emptyset$

$$0 \to \mathbb{Z}_2^2 \xrightarrow{\begin{pmatrix}1 & 1\end{pmatrix}} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to 0$$

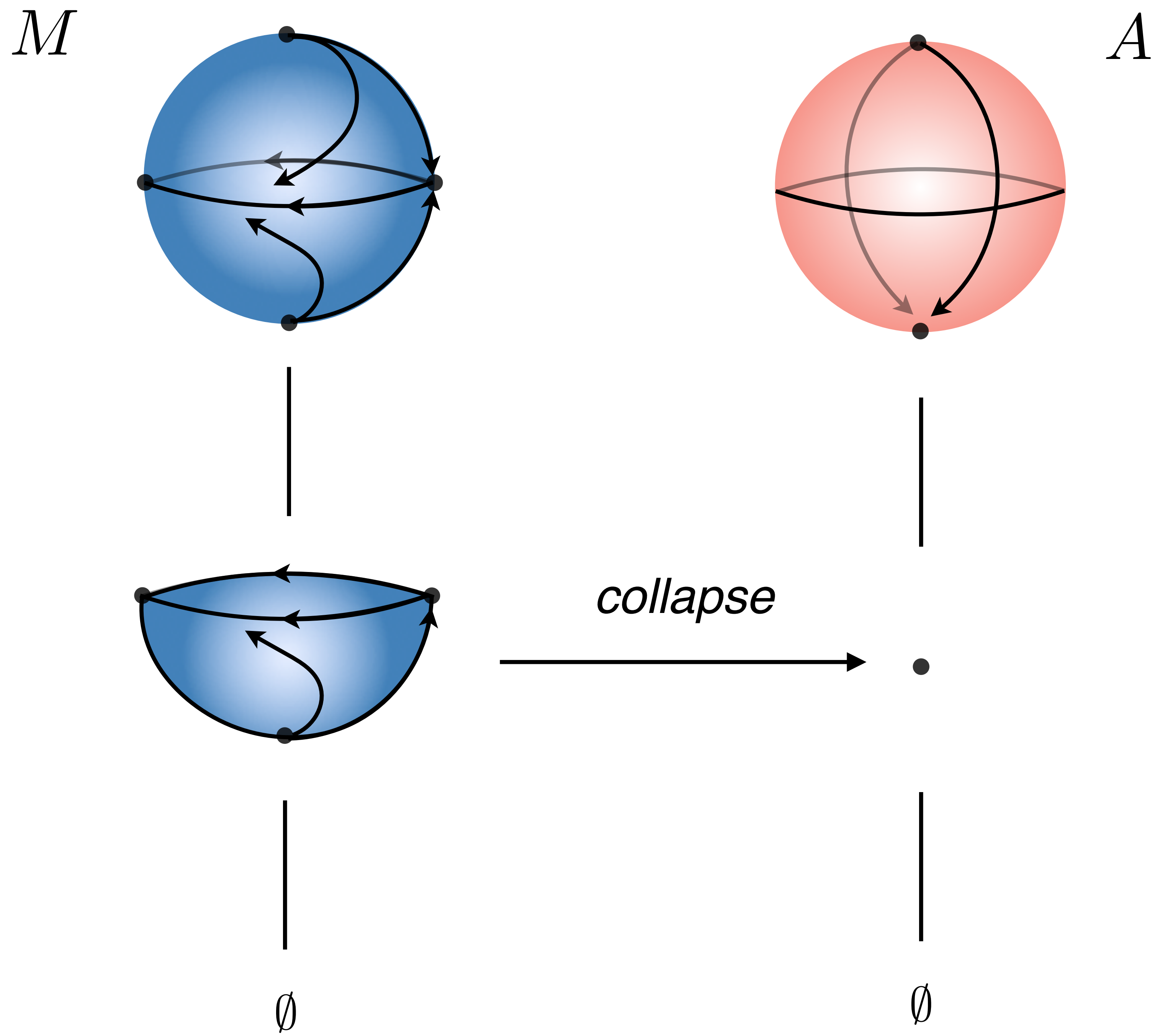
$$0 \to \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to 0$$

$$0$$

choose a sublattice of lattice of attractors

filtered complex may be large - replace with smaller representation

Conley data reduction

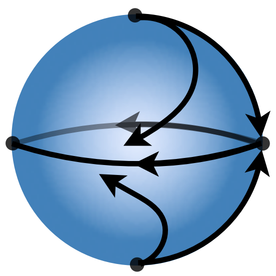


filtered complex

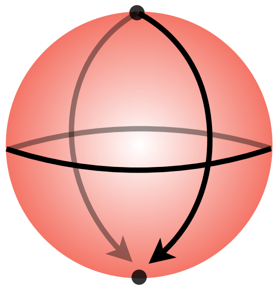
reduced representation

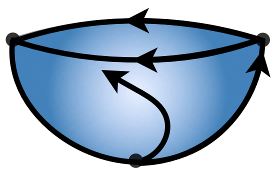
Conley data reduction II

$$M \longrightarrow A$$



$$0 \to \mathbb{Z}_2^2 \xrightarrow{\begin{pmatrix}1 & 1\end{pmatrix}} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to 0 \qquad\qquad 0 \to \mathbb{Z}_2 \to 0 \to \mathbb{Z}_2 \to 0$$





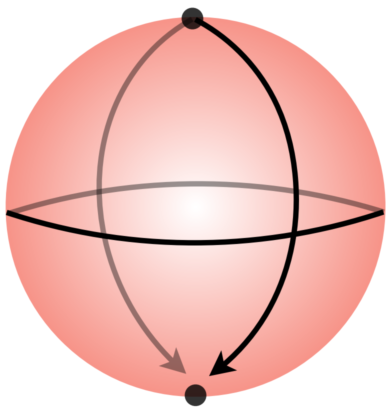
$$0 \to \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to 0 \longrightarrow 0 \to 0 \to 0 \to \mathbb{Z}_2 \to 0$$

$\emptyset$ $\qquad 0 \qquad\qquad 0 \qquad$ $\emptyset$

filtered complex $\longrightarrow$ reduced filtered complex

filtered homotopy equivalence

connection matrix



A

$$0 \to \mathbb{Z}_2 \to 0 \to \mathbb{Z}_2 \to 0$$

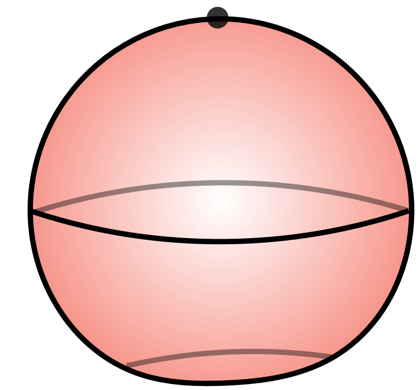
A_β

$$0 \to 0 \to 0 \to \mathbb{Z}_2 \to 0$$

$\emptyset$ $\quad 0 \quad A_0$

A/A_β

$$0 \to \mathbb{Z}_2 \to 0 \to \underset{0}{\cancel{\mathbb{Z}_2}} \to 0$$



$$A/A_\beta = H(A/A_\beta)$$

Conley index

in general…

(A, L) is a 'connection matrix' if

$$p \in J(L) \quad \partial(A_p) \subset A_{Pred(p)}$$

equivalently..

$$A_p/A_{Pred(p)} = H(A_p/A_{Pred(p)})$$

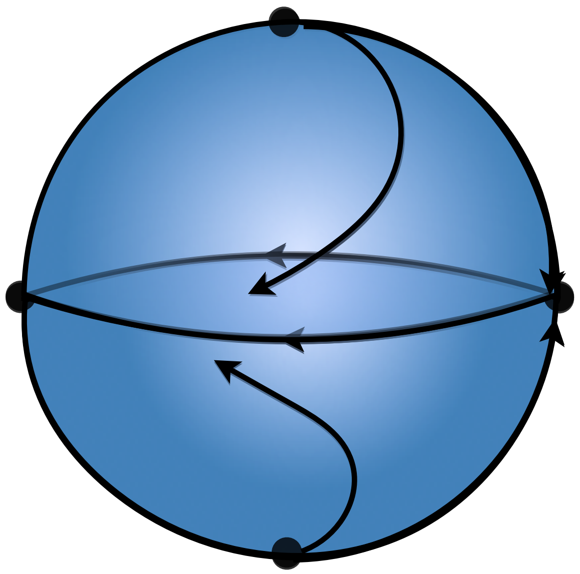
connection matrix is to filtered complex as homology is to chain complex

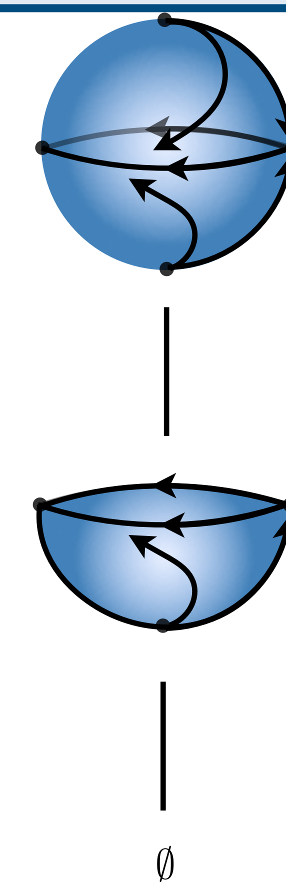
Theorem: this viewpoint is equivalent to Franzosa's construction

computation

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{chain complex} \arrow[r, dash, "\textit{dictionary}"] \arrow[d, "\textit{discrete Morse theory}"'] & \text{lattice-filtered complex} \arrow[d, "\textit{decompose into quotients}"', "\textit{discrete Morse theory on the quotients}"] \\
\text{homology} \arrow[r, dash] & \text{connection matrix}
\end{tikzcd}$$

J(L)-decomposition





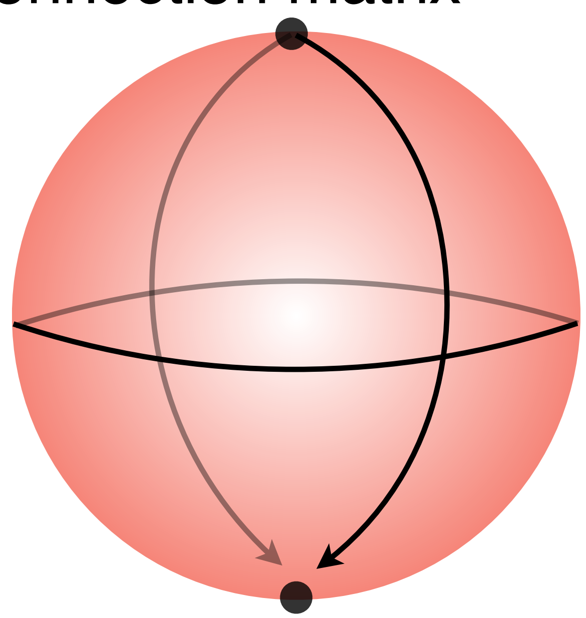
$$M \qquad = \qquad M/M_\beta \quad \oplus \quad M_\beta$$

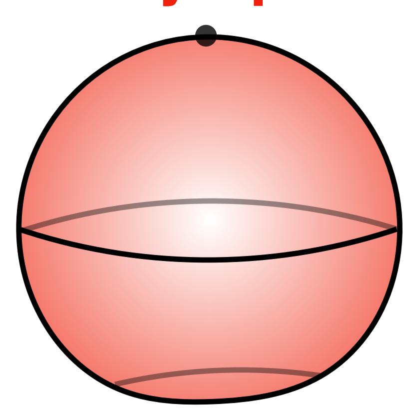
Theorem: lattice filtered complex has splitting $M = \oplus_{p\in J(L)} M_p/M_{Pred(p)}$

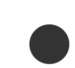
connection matrix

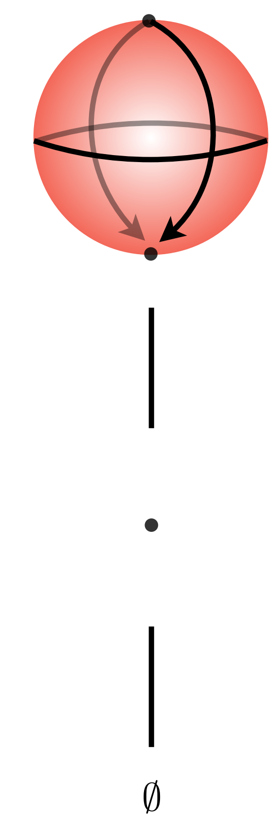
$A/A_\beta = H(M/M_\beta)$ $\qquad$ $A_\beta = H(M_\beta)$

boundary operator on Conley indices





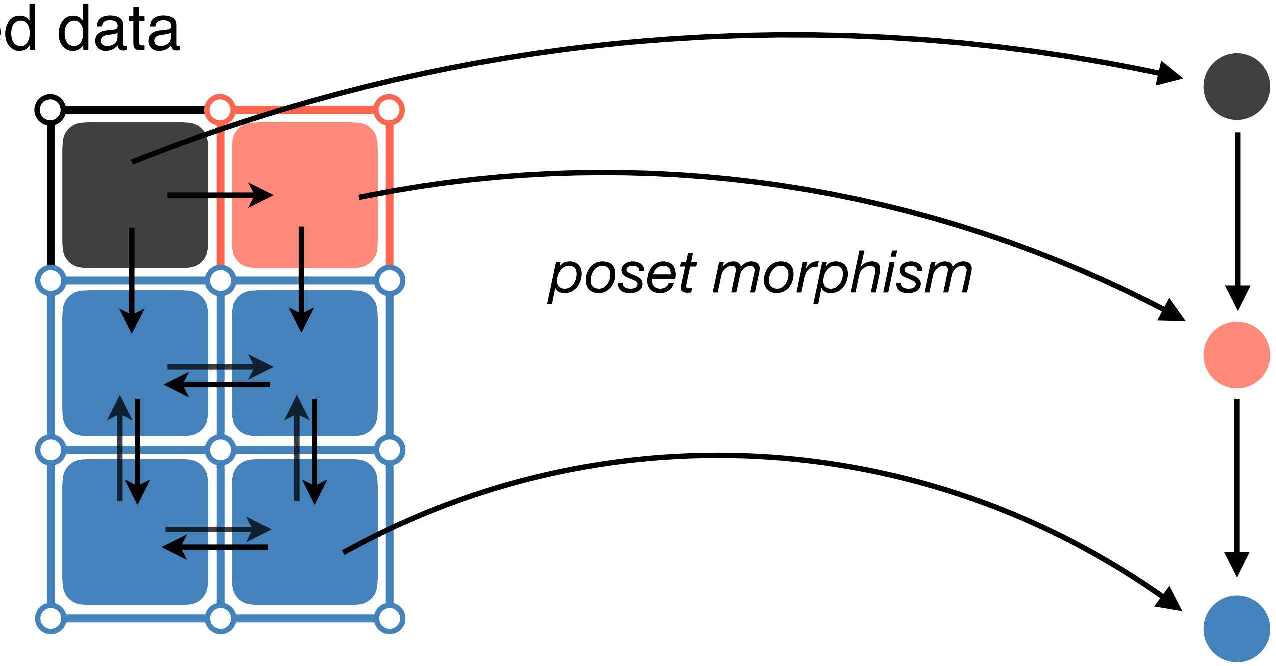




$$A \qquad = \qquad A/A_\beta \quad \oplus \quad A_\beta$$

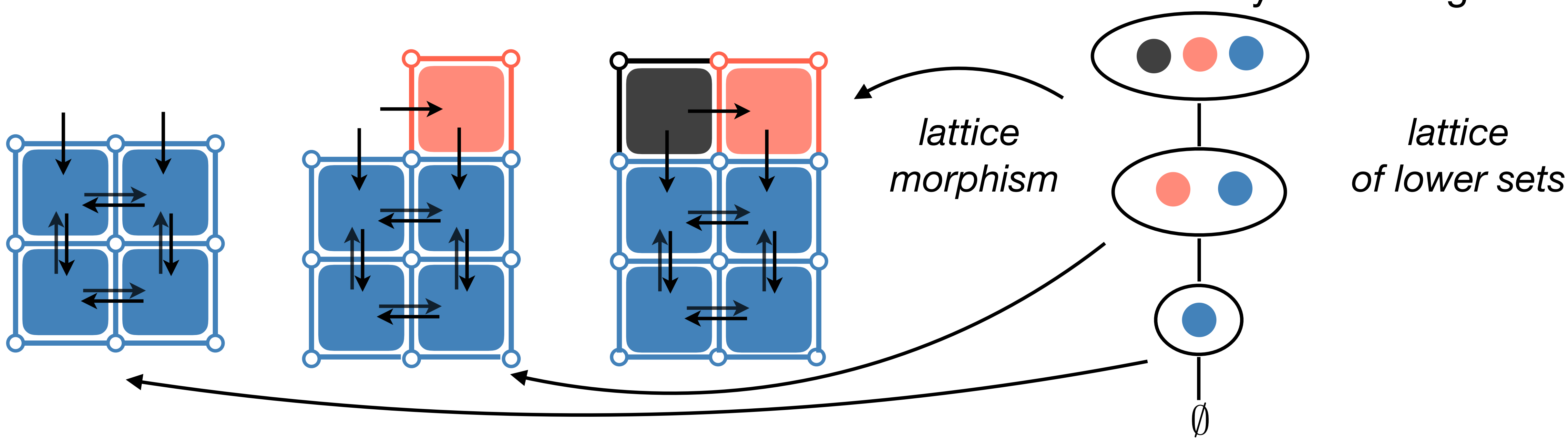
data structure

dynamics provides the filtered data



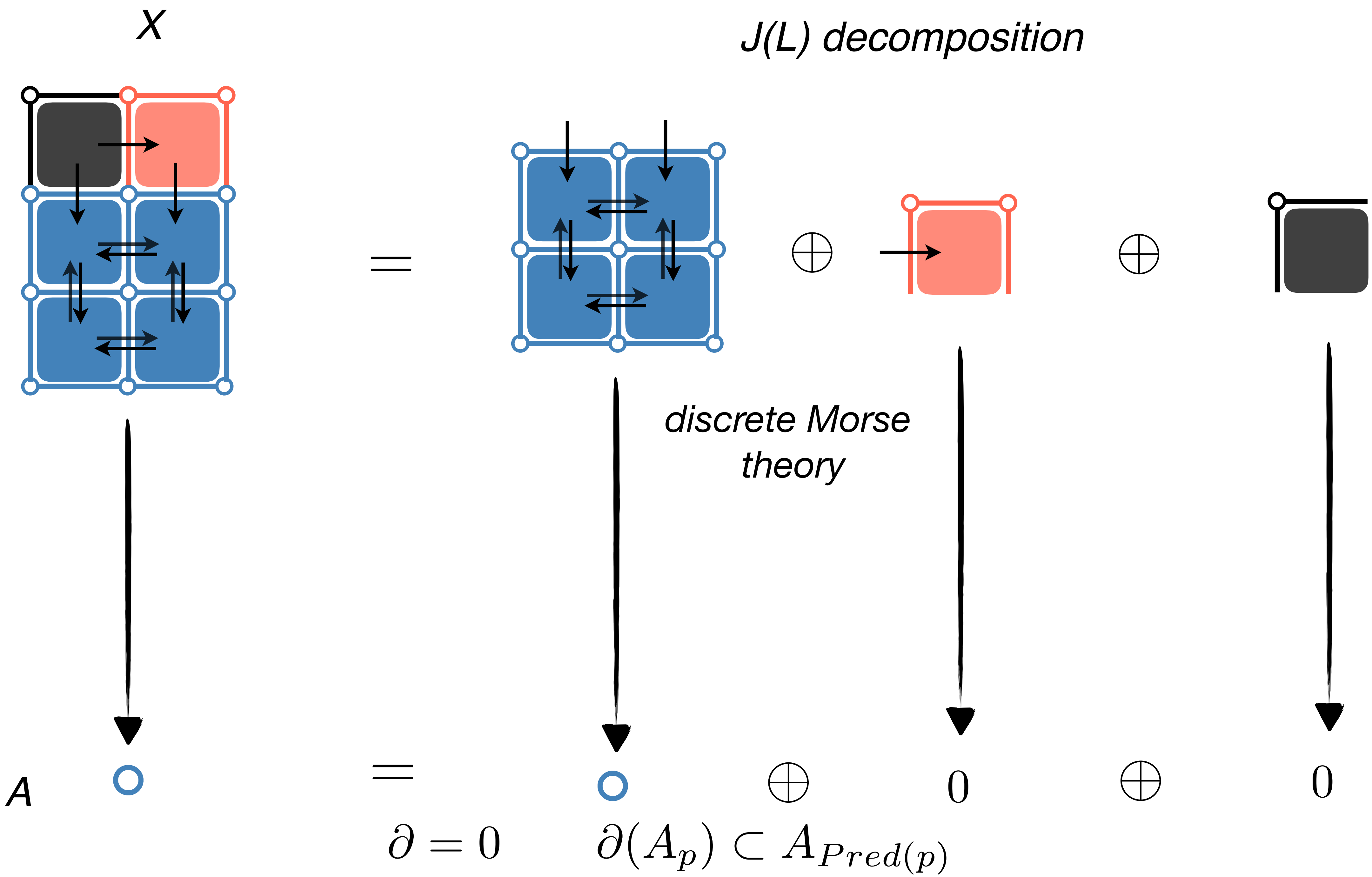
cell complex, filtered with order preserving map to poset*

…Birkhoff's Theorem says we can go back



*equivalently: the pre-image of lower sets form lower sets (continuous in *Alexandroff topology*)

(filtered) discrete Morse theory



Theorem: filtered discrete Morse theory produces a connection matrix

Remark: the boundary operator of the reduced filtered complex is a connection matrix a la Franzosa thank you for your attention

Collaborators:
S. Harker
K. Mischaikow