A Computational Framework for Connection Matrices

…toward a computational homological theory of dynamics

Kelly Spendlove, Rutgers University

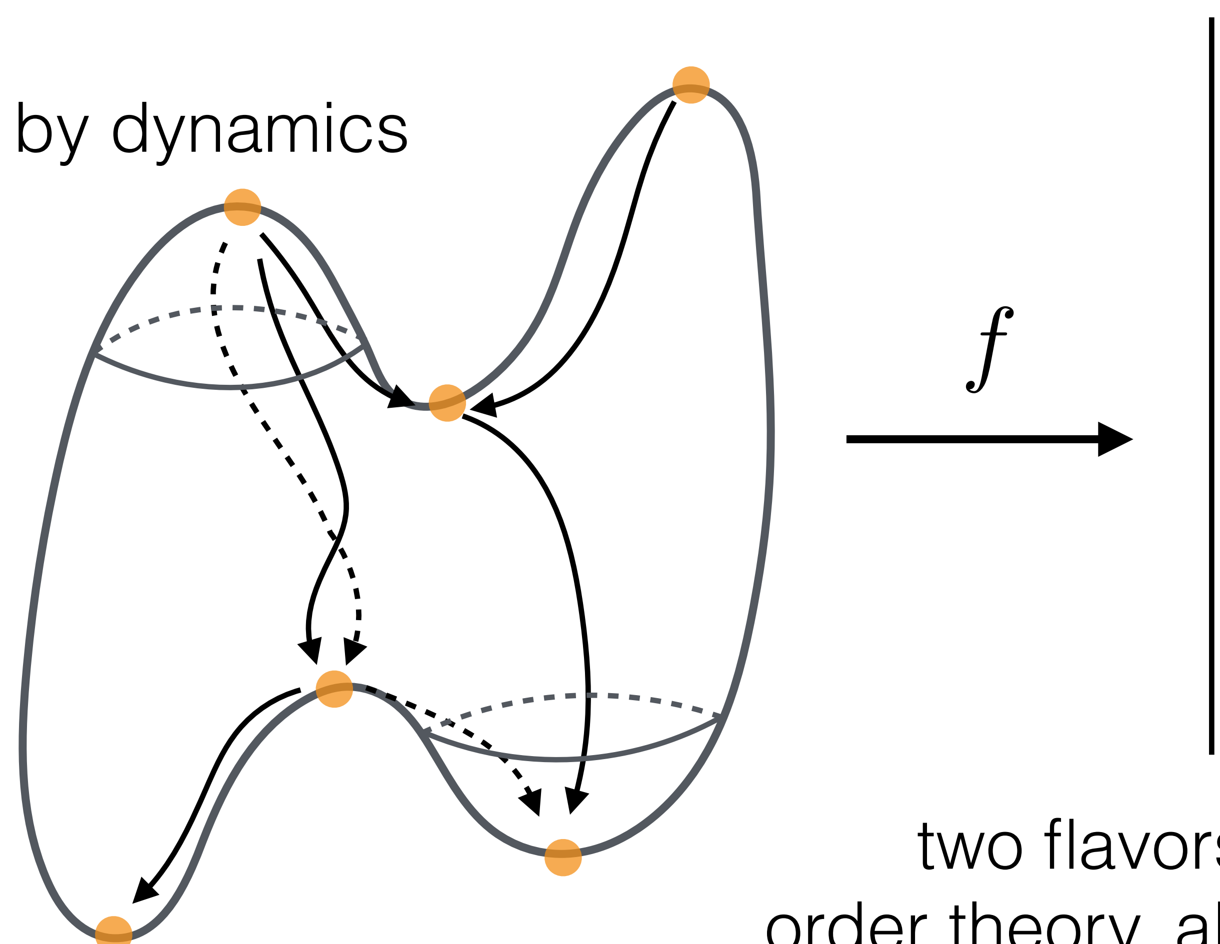
ATDD, 2018

dynamical musings

- a dynamical system engenders topological data
- local data (e.g. equilibria) and global data (attractors)
- topological data are ordered and measured with algebra

data are *ordered* by dynamics

$$\dot{x} = -\nabla f$$



data have *algebraic invariants* (homology)

two flavors of algebra:
order theory, algebraic topology

Conley-Morse Theory

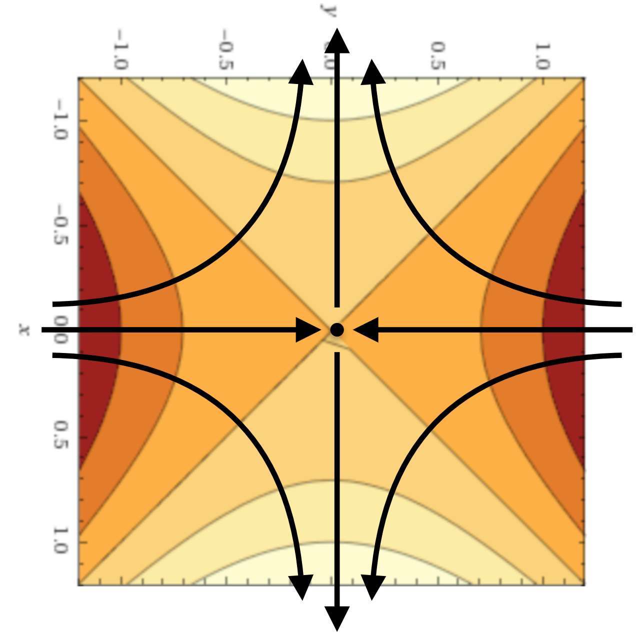
'…if such rough equations are to be of use it is necessary to study them in rough terms.'

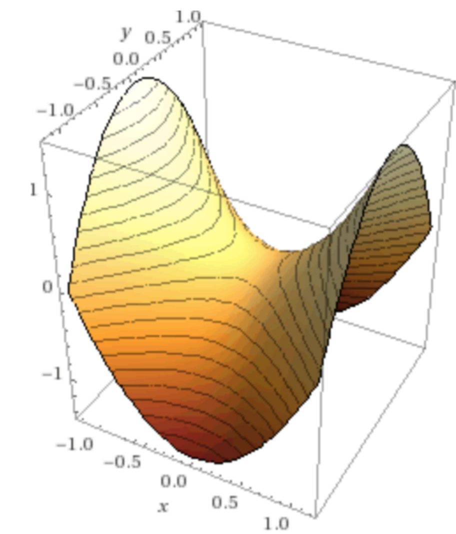
C. Conley, CBMS Monograph (1978)

Morse indices measure fixed points

Morse index quantifies instability

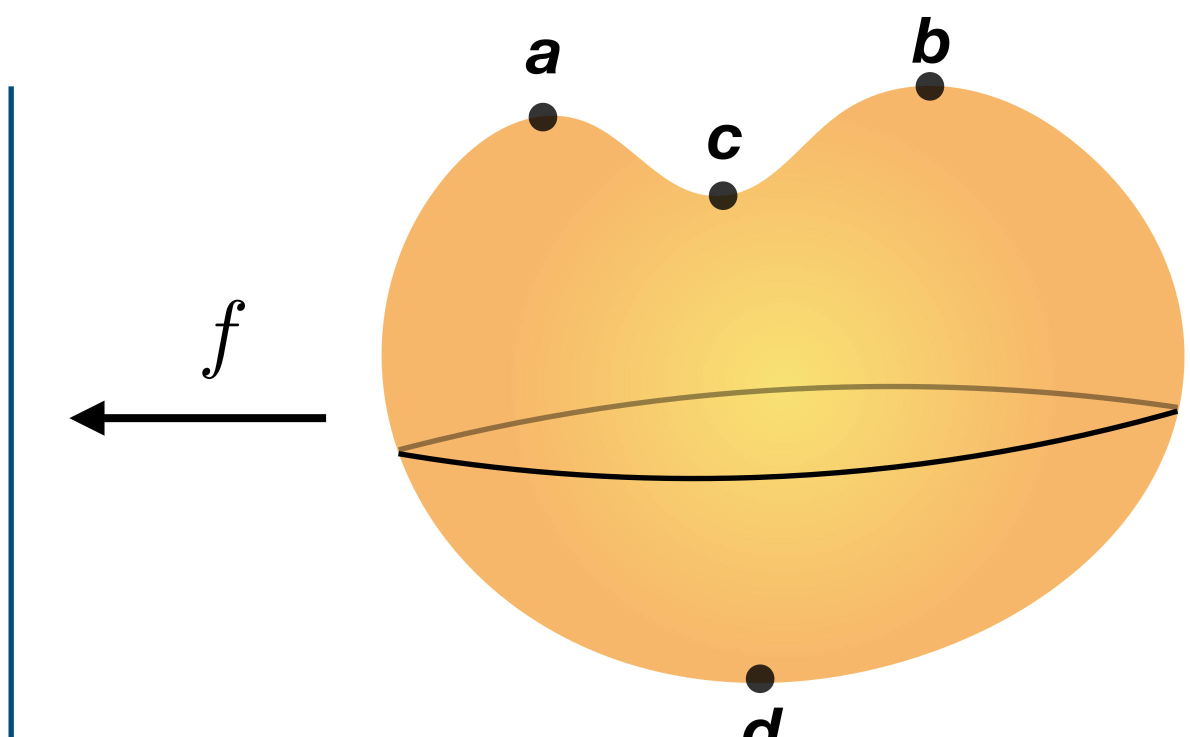
dimension of $W^u(p)$





Morse indices as cyclic chain complex (zero differentials)

$$M_n(p) = \begin{cases} \mathbb{Z}_2\langle p\rangle, & n = \dim W^u(p) \\ 0, & \text{else} \end{cases}$$



$$\dot{x} = -\nabla f(x)$$

Morse indices assemble

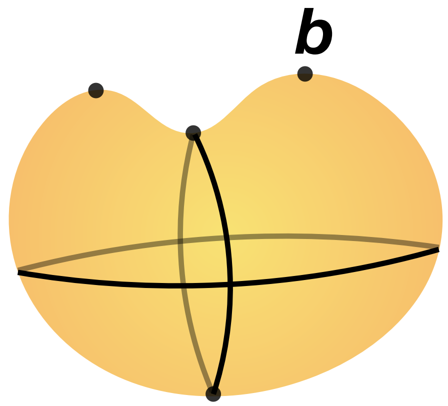
$$M_\bullet(f) = \bigoplus_{p \in crit(f)} M_\bullet(p)$$

$$\begin{array}{c} 0 \\ \downarrow \\ \mathbb{Z}_2\langle a\rangle \oplus \mathbb{Z}_2\langle b\rangle \\ \downarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \\ \mathbb{Z}_2\langle c\rangle \\ \downarrow 0 \\ \mathbb{Z}_2\langle d\rangle \\ \downarrow \\ 0 \end{array}$$

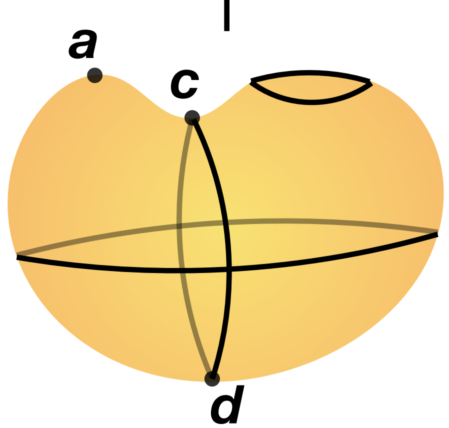
boundary operator counts connecting orbits mod 2

a height function filters

(via lattice of sublevel sets)



$$\longrightarrow \quad 0 \leftarrow \mathbb{Z}_2\langle d\rangle \xleftarrow{0} \mathbb{Z}_2\langle c\rangle \xleftarrow{\begin{pmatrix}1 & 1\end{pmatrix}} \mathbb{Z}_2\langle a\rangle \oplus \mathbb{Z}_2\langle b\rangle \leftarrow 0$$



$$\longrightarrow \quad 0 \leftarrow \mathbb{Z}_2\langle d\rangle \xleftarrow{0} \mathbb{Z}_2\langle c\rangle \xleftarrow{1} \mathbb{Z}_2\langle a\rangle \leftarrow 0$$

simple dynamics:
non-degenerate equilibria
heteroclinic orbits

$$f^{-1}(-\infty, x] \rightsquigarrow \{\mathbb{Z}_2\langle a\rangle : f(a) \leq f(x)\}$$

sublevel set (Morse) subcomplex

Morse index of ***b*** recovered
as a subquotient

$$M_\bullet(b) = \frac{0 \leftarrow \mathbb{Z}_2\langle d\rangle \xleftarrow{0} \mathbb{Z}_2\langle c\rangle \xleftarrow{\begin{pmatrix}1 & 1\end{pmatrix}} \mathbb{Z}_2\langle a\rangle \oplus \mathbb{Z}_2\langle b\rangle \leftarrow 0}{0 \leftarrow \mathbb{Z}_2\langle d\rangle \xleftarrow{0} \mathbb{Z}_2\langle c\rangle \xleftarrow{1} \mathbb{Z}_2\langle a\rangle \leftarrow 0}$$

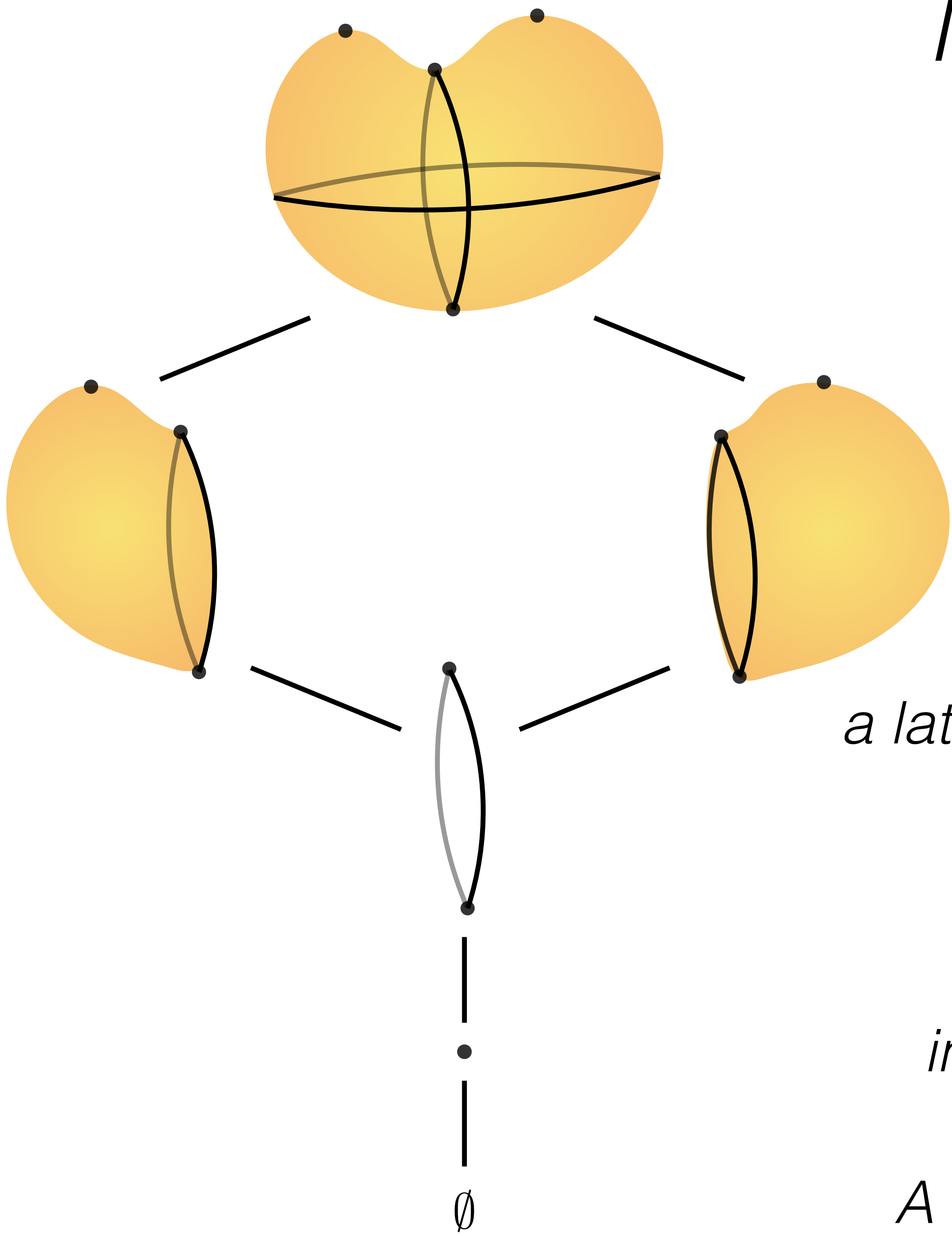
$$0 \leftarrow 0 \leftarrow 0 \leftarrow \mathbb{Z}_2\langle b\rangle \leftarrow 0 = M_\bullet(b)$$

Conley's focus: attractors

Conley theory is a purely topological generalization of Morse theory
for general dynamical systems

A is an attractor if there is a neighborhood *N* of *A* with **ω**(*N*)=*A*

$$\omega(N) = \bigcap_{s\in\mathbb{R}} \overline{\{\varphi(N,t) : t > s\}}$$



attractors have order structure

$\wedge := \cap \quad \vee := \cup$

a lattice of attractors

in picture: lattice of attractors
in practice: lattice of attracting blocks

A is an attracting block if $\varphi(A,t) \subset int(A)$ *for all t > 0*

Birkhoff's theorem

L finite distributive lattice

the poset of *join irreducible* elements of **L** is

$$\mathsf{J}(\mathsf{L}) := \{x \in \mathsf{L}\backslash\{0_L\} : \text{ if } x = a \vee b, \text{ then } a = x \text{ or } b = x\}$$

a join-irreducible has a unique predecessor

$Pred : \mathsf{J}(\mathsf{L}) \to \mathsf{L}$

$(\mathsf{P}, \leq)$ poset

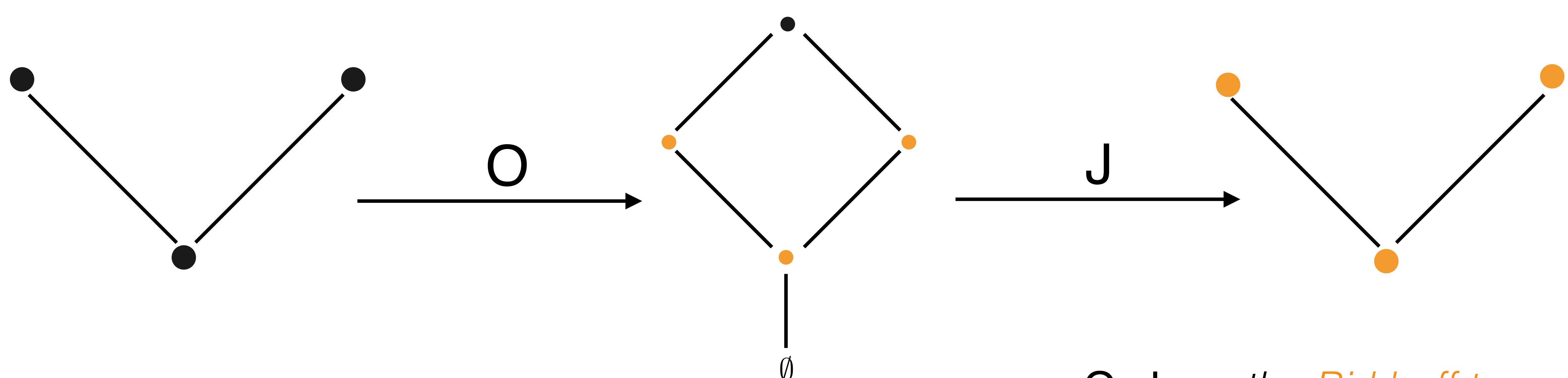
the lattice of lower sets is

$$\mathsf{O}(\mathsf{P}) := \{U \subseteq \mathsf{P} : \text{ if } x \in U \text{ and } y \leq x \text{ then } y \in U\}$$

$\wedge := \cap \quad \vee := \cup$

Fact: **O**, **J** are contravariant functors

Birkhoff: $\mathsf{O}(\mathsf{J}(\mathsf{L})) \cong \mathsf{L} \qquad \mathsf{J}(\mathsf{O}(\mathsf{P})) \cong \mathsf{P}$



O, J *are the Birkhoff transforms*

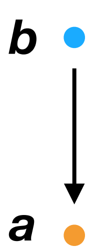
Conley-Morse Homology

to generalize Morse homology

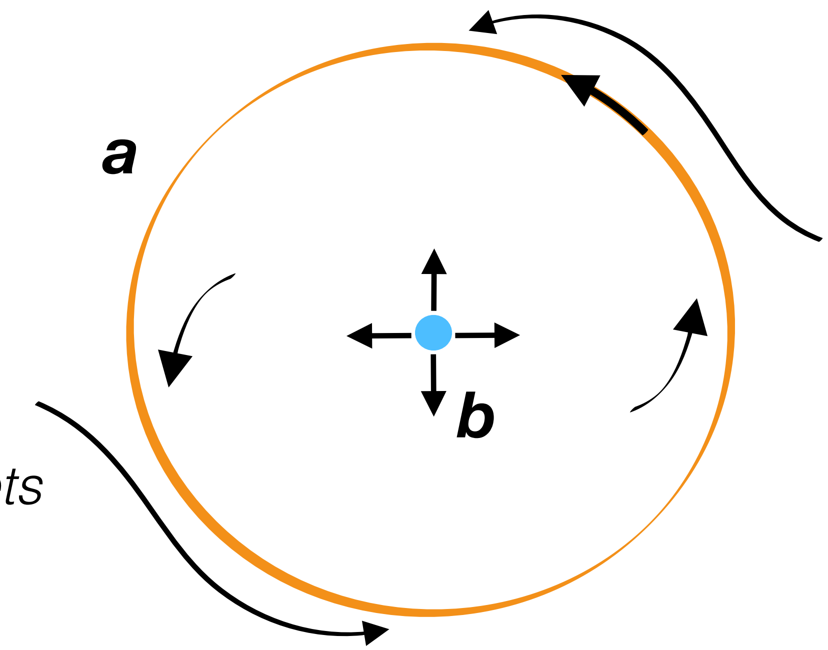
associate cyclic complex to isolated invariant sets (Conley index)

characterized by dynamics at the boundary (local instability)

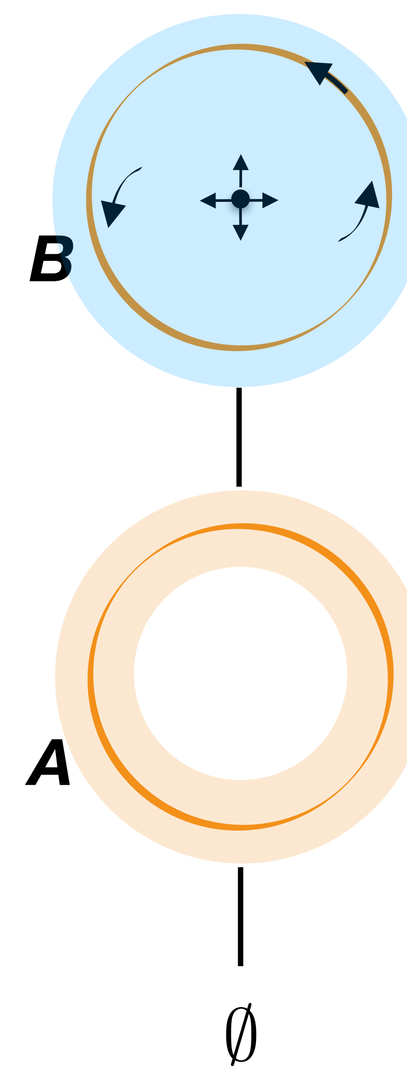
$$CH_\bullet(b) = H_\bullet(B, Pred(B)) \quad B = \mu(b)$$



poset P
of invariant sets



$$P \xrightarrow[\cong]{\mu} J(L)$$
$$b \longmapsto B$$
$$a \longmapsto A$$



L *lattice of attracting blocks*

chain complex of Conley indices

$$0 \leftarrow \mathbb{Z}_2\langle a\rangle \xleftarrow{0} \mathbb{Z}_2\langle a\rangle \xleftarrow{1} \mathbb{Z}_2\langle b\rangle \leftarrow 0$$

boundary operator is called the connection matrix

Franzosa, Mischaikow, McCord, Reineck… Conley-Morse homology is a homology theory

Conley-Morse Homology

to generalize Morse homology

Conley indices as input to chain complex

what is the boundary operator?

for **L** lattice of attracting blocks and **J(L)** join-irreducibles

Theorem (Franzosa, Robbin & Salamon): There exists a strictly upper triangular - wrt $(\mathsf{J}(\mathsf{L}), \leq)$- boundary operator

$$\Delta : \bigoplus_{p \in \mathsf{J}(\mathsf{L})} CH_\bullet(p) \to \bigoplus_{p \in \mathsf{J}(\mathsf{L})} CH_\bullet(p)$$

so that for any attracting block *A* in **L** the induced homology

local to global

$$\Delta : \bigoplus_{p \in A} CH_\bullet(p) \to \bigoplus_{p \in A} CH_\bullet(p)$$

is isomorphic to $H_\bullet(A)$

algebraic representation of dynamics

Δ is called a connection matrix

caveat: chain complex braids, graded module braids

Categories + Data Structures

‘data! data! data! I can’t make bricks without clay.’
S. Holmes, The Adventure of the Copper Beaches (1892)

L finite, distributive lattice (C, ∂) chain complex

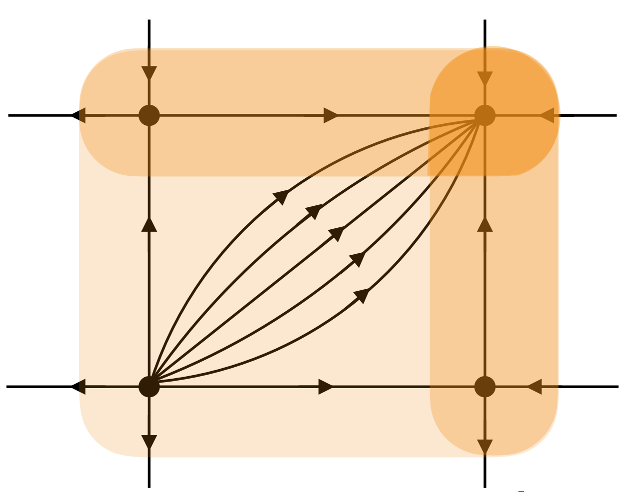
Definition (L-filtered chain complex)

(C, ∂), L and lattice homomorphism from L to the (modular) lattice of subcomplexes of (C, ∂)

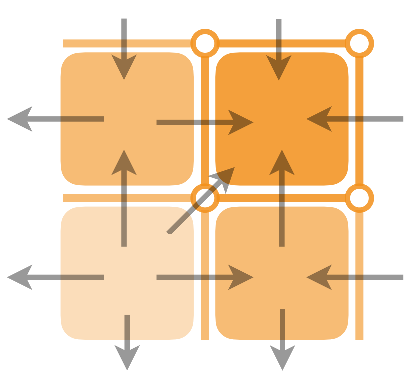
$$\mathsf{L} \longrightarrow \mathsf{Sub}(C, \partial)$$

for the talk we'll write $\{C_\bullet^a\}_{a \in \mathsf{L}}$

computational dynamics *homological algebra*



attracting block $\longrightarrow$ *subcomplex*



in practice:

L *comes from multi-valued map or outer approximation*

(C, ∂) *is a cell complex with basis*

Kalies, Mischaikow, van der Vorst, Mrozek, …

X cellular complex
(Lefschetz, CW)

L finite, distributive lattice

$(X, \leq)$ face poset

J(L) *poset of join-irreducibles*

Definition (J(L)-graded cell complex)

X, J(L), and a poset morphism ν from X to J(L)

$$(X, \leq) \xrightarrow{\nu} (J(L), \leq)$$

Birkhoff transform gives filtered complex

$$L \xrightarrow{O(\nu)} \mathrm{Sub}(X)$$

category $\boldsymbol{Ch}(L)$ of **L**-filtered chain complexes

homotopy category $\boldsymbol{K}(L)$ of **L**-filtered chain complexes

interpretation of connection matrix for data analysis:
‘small’ representative of homotopy equivalence class

moral: homotopy categories for chain-level data reduction without loss of homological information

the category of **L**-filtered chain complexes

a map ϕ is filtered if

$$\phi(C^a_\bullet) \subseteq M^a_\bullet$$

$$
\begin{tikzcd}
 & C_\bullet \arrow[dl, dash] \arrow[dr, dash] \arrow[rrrr, "\phi"] & & & & M_\bullet \arrow[dl, dash] \arrow[dr, dash] & \\
C^a_\bullet \arrow[dr, dash] \arrow[rrrr, bend left] & & C^b_\bullet \arrow[dl, dash] \arrow[rrrr, bend left] & & M^a_\bullet \arrow[dr, dash] & & M^b_\bullet \arrow[dl, dash] \\
 & C^c_\bullet \arrow[d, dash] \arrow[rrrr, bend left] & & & & M^c_\bullet \arrow[d, dash] & \\
 & 0 \arrow[rrrr] & & & & 0 &
\end{tikzcd}
$$

in ***Ch**(L)* objects are filtered complexes, morphisms filtered chain maps

the homotopy category for *L*-filtered chain complexes

Definition (Filtered homotopy equivalence)

$$\begin{tikzcd} \{C^a_\bullet\}_{a\in \mathsf{L}} \arrow[loop above, "h"] \arrow[r, shift left, "\psi"] & \{M^a_\bullet\}_{a\in \mathsf{L}} \arrow[loop above, "h'"] \arrow[l, shift left, "\phi"] \end{tikzcd}$$

Quadruple (ψ, ϕ, h, h') such that

$$\psi \circ \phi - id_C = h\partial^C + \partial^C h$$
$$\phi \circ \psi - id_M = h'\partial^M + \partial^M h'$$

ψ, ϕ *are filtered chain maps* h, h' *are filtered homotopies*

objects in ***K****(****L****) are filtered complexes and morphisms are homotopy equivalence classes*

isomorphisms in ***K****(****L****) are filtered homotopy equivalences*

Definition (*Conley filtered*) (connection matrix for data analysis)

$$\{C_\bullet^a\}_{a\in\mathsf{L}} \text{ such that } \partial(C_\bullet^q) \subseteq C_\bullet^{Pred(q)} \text{ for } q \in \mathsf{J}(\mathsf{L})$$

Proposition: Over fields, any filtered complex admits a J(L)-splitting

$$C = \bigoplus_{q\in\mathsf{J}(\mathsf{L})} M^q \quad \text{where } M^q \cong C^q/C^{Pred(q)} \qquad \partial : \bigoplus_{q\in\mathsf{J}(\mathsf{L})} M^q \to \bigoplus_{q\in\mathsf{J}(\mathsf{L})} M^q$$

A subspace M^q *corresponds to a invariant set*

the (p,q) entry $\partial^{p,q} : M^q \to M^p$ *corresponds to connecting orbits*

J(L)-splitting for Conley filterings

$$\Delta : \bigoplus_{q\in\mathsf{J}(\mathsf{L})} H_\bullet(C^q, C^{Pred(q)}) \to \bigoplus_{q\in\mathsf{J}(\mathsf{L})} H_\bullet(C^q, C^{Pred(q)})$$

this is the classical formula of Franzosa

Robbin + Salamon, Harker + Mischaikow + S.

Framework for Connection Matrices

$$
\begin{tikzcd}[column sep=huge]
\textit{chain complex} \arrow[d, shift left] \arrow[r, dash, "\textit{dictionary}"] & \textit{filtered chain complex} \arrow[d, shift right] \\
\textit{homology} \arrow[u, shift left] \arrow[r, dash] & \textit{Conley-filtered complex} \arrow[u, shift right]
\end{tikzcd}
$$

reductions

$$\begin{tikzcd} (C_\bullet, \partial^C) \arrow[loop left, "h"] \arrow[r, shift left, "\psi"] & (M_\bullet, \partial^M) \arrow[l, shift left, "\phi"] \end{tikzcd}$$

diagram of chain complexes

ϕ, ψ chain maps

h homotopy

$$\psi \circ \phi = id_M$$

$$\phi \circ \psi = id_C + \partial^C \circ h + h \circ \partial^C$$

$$h \circ h = 0, \psi \circ h = 0, h \circ \phi = 0$$

consequence of the first identity:

ϕ is a injective and ψ is an surjective

M is called the *reduced complex* (want this much smaller)

from the first two identities:

a reduction is a special type of homotopy equivalence

a homotopy equivalence induces isomorphisms on homology $H_\bullet(C_\bullet) \cong H_\bullet(M_\bullet)$

reductions: Homological perturbation theory, effective homology theory, ….

reductions

$$\begin{tikzcd} (C_\bullet, \partial^C) \arrow[loop left, "h"] \arrow[r, shift left, "\psi"] & (M_\bullet, \partial^M) \arrow[l, shift left, "\phi"] \end{tikzcd}$$

diagram of chain complexes

ϕ, ψ chain maps

h homotopy

$$\psi \circ \phi = id_M$$

$$\phi \circ \psi = id_C + \partial^C \circ h + h \circ \partial^C$$

$$h \circ h = 0, \psi \circ h = 0, h \circ \phi = 0$$

using the third set of identities: $C_\bullet = M_\bullet \oplus \ker \psi$

$\ker \psi$ is acyclic, i.e. $H_\bullet(\ker \psi) = 0$

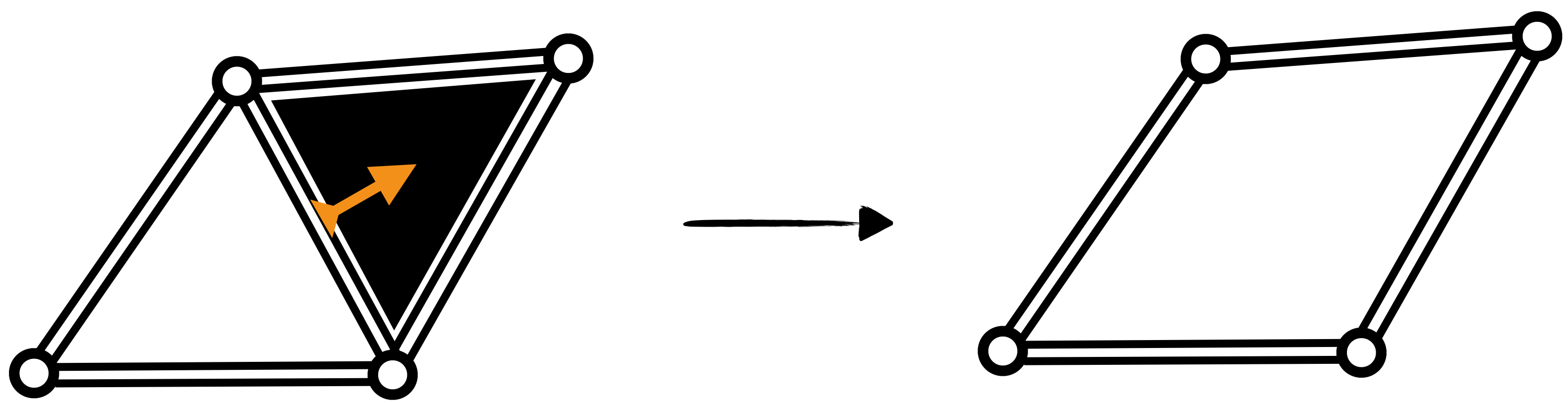
$$H_\bullet(C_\bullet) \cong H_\bullet(M_\bullet)$$

… can be done in any category (e.g. filtered) of chain complexes in practice, $(C_\bullet, \partial)$ is cellular and has distinguished basis

discrete Morse theory operates on a basis to simplify complexes

write the basis as disjoint union of critical and paired $\{c_i\} \sqcup \{\sigma_i \leq \tau_i\}$

incidence number of pair must be a unit



Proposition: Discrete Morse theory gives rise to reduction

critical cells

$$\begin{tikzcd} (C_\bullet, \partial^C) \arrow[loop left, "h"] \arrow[r, shift left, "\psi"] & (M_\bullet, \partial^M) \arrow[l, shift left, "\phi"] \end{tikzcd}$$

$\ker \psi$ (paired cells)

Morse complex

Forman, Harker, Sergeraert, Rubio, Nanda…

tower of reductions

$$\begin{tikzcd} C_\bullet \arrow[loop left, "h"] \arrow[r, shift left, "\psi^0"] & M^0_\bullet \arrow[l, shift left, "\phi^0"] \arrow[loop above, "h^0"] \arrow[r, shift left, "\psi^1"] & \cdots \arrow[l, shift left, "\phi^1"] \arrow[r, shift left, "\psi^n"] & M^n_\bullet \arrow[l, shift left, "\phi^n"] \end{tikzcd}$$

Iterated discrete Morse theory leads to tower of reductions

Proposition: After a finite number of applications of discrete Morse theory the tower stabilizes with $\partial^{M^n} = 0$

(over a field)

$$\partial^{M^n} = 0 \implies M^n_\bullet = H_\bullet(M^n)$$

Corollary: Homology may be computed with discrete Morse theory

Nanda, Dlotko + Wagner, Harker + Mischaikow + S.

reductions of filtered chain complexes

$$\begin{tikzcd} \{C^a_\bullet\}_{a\in\mathsf{L}} \arrow[loop above] \arrow[r, shift left] & \{M^a_\bullet\}_{a\in\mathsf{L}} \arrow[l, shift left] \end{tikzcd}$$

*diagram in **Ch**(L)*

Proposition: Filtered Morse pairing gives rise to filtered reduction

a filtered pairing only pairs cells associated with the same join-irreducible

$$\begin{tikzcd} \{C^a_\bullet\}_{a\in\mathsf{L}} \arrow[loop above] \arrow[r, shift left] & \{M^a_\bullet\}^0_{a\in\mathsf{L}} \arrow[loop above] \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & \{M^a_\bullet\}^n_{a\in\mathsf{L}} \arrow[l, shift left] \end{tikzcd}$$

Proposition: After a finite number of applications of filtered discrete Morse theory the tower stabilizes with $\{M^a_\bullet\}^n_{a\in\mathsf{L}}$ Conley-filtered

(over a field)

Harker + Mischaikow + S.

relationship to persistence

when L *is a total order*

$$\begin{tikzcd} \{C^a_\bullet\}_{a\in \mathsf{L}} \arrow[loop above] \arrow[r, shift left] & \{M^a_\bullet\}_{a\in \mathsf{L}} \arrow[l, shift left] \end{tikzcd}$$

reduction between filtrations

$$\begin{tikzcd} C_\bullet \arrow[loop left] \arrow[r, shift left, "\psi"] & M_\bullet \arrow[l, shift left, "\phi"] \\ \vdots \arrow[u, dash] & \vdots \arrow[u, dash] \\ C^b_\bullet \arrow[u, dash] \arrow[r, shift left] & M^b_\bullet \arrow[u, dash] \arrow[l, shift left] \\ C^a_\bullet \arrow[u, dash] \arrow[r, shift left] & M^a_\bullet \arrow[u, dash] \arrow[l, shift left] \\ 0 \arrow[u, dash] \arrow[r, shift left] & 0 \arrow[u, dash] \arrow[l, shift left] \end{tikzcd}$$

Theorem: For any reduction, the persistent homology groups for $\{C^a_\bullet\}_{a\in \mathsf{L}}$ and $\{M^a_\bullet\}_{a\in \mathsf{L}}$ are canonically isomorphic

$$C_\bullet = M_\bullet \oplus \ker\psi$$

$\ker\psi$ *contains pairs with zero persistence*

Theorem: If $\{M^a_\bullet\}_{a\in \mathsf{L}}$ is Conley filtered, it is the smallest complex (up to isomorphism) for computing persistent homology

Remark: Such reductions are the beginning of a spectral sequence-type algorithm (Edelsbrunner & Harer, Bauer et al, ...)

Harker + Mischaikow + S.

computational Conley homology

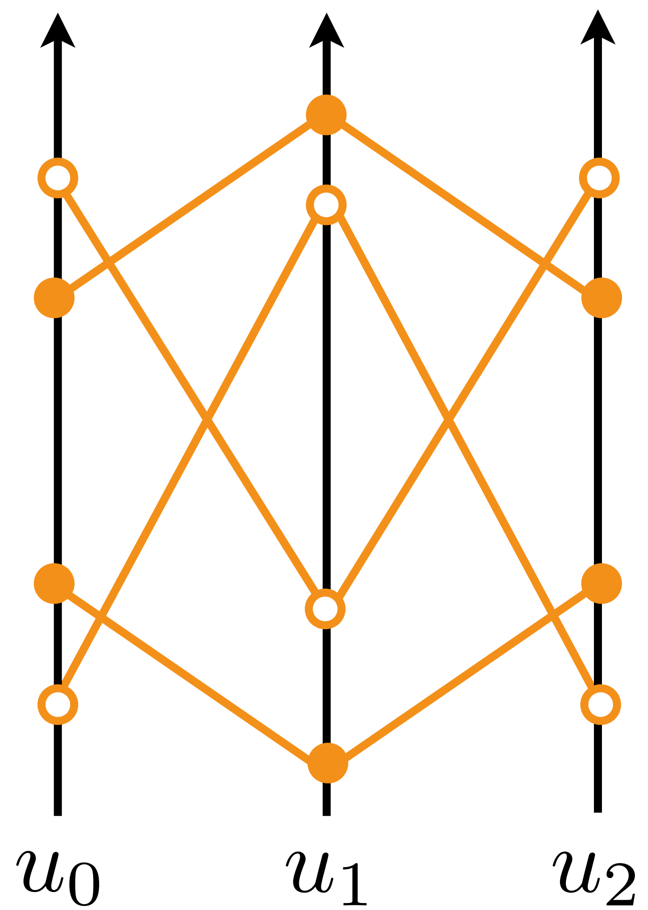
application + implementation + pedagogy

‘…it is the author’s belief that in its present form the connection matrix can be applied to many interesting problems by individuals with little or no training in algebraic topology.’

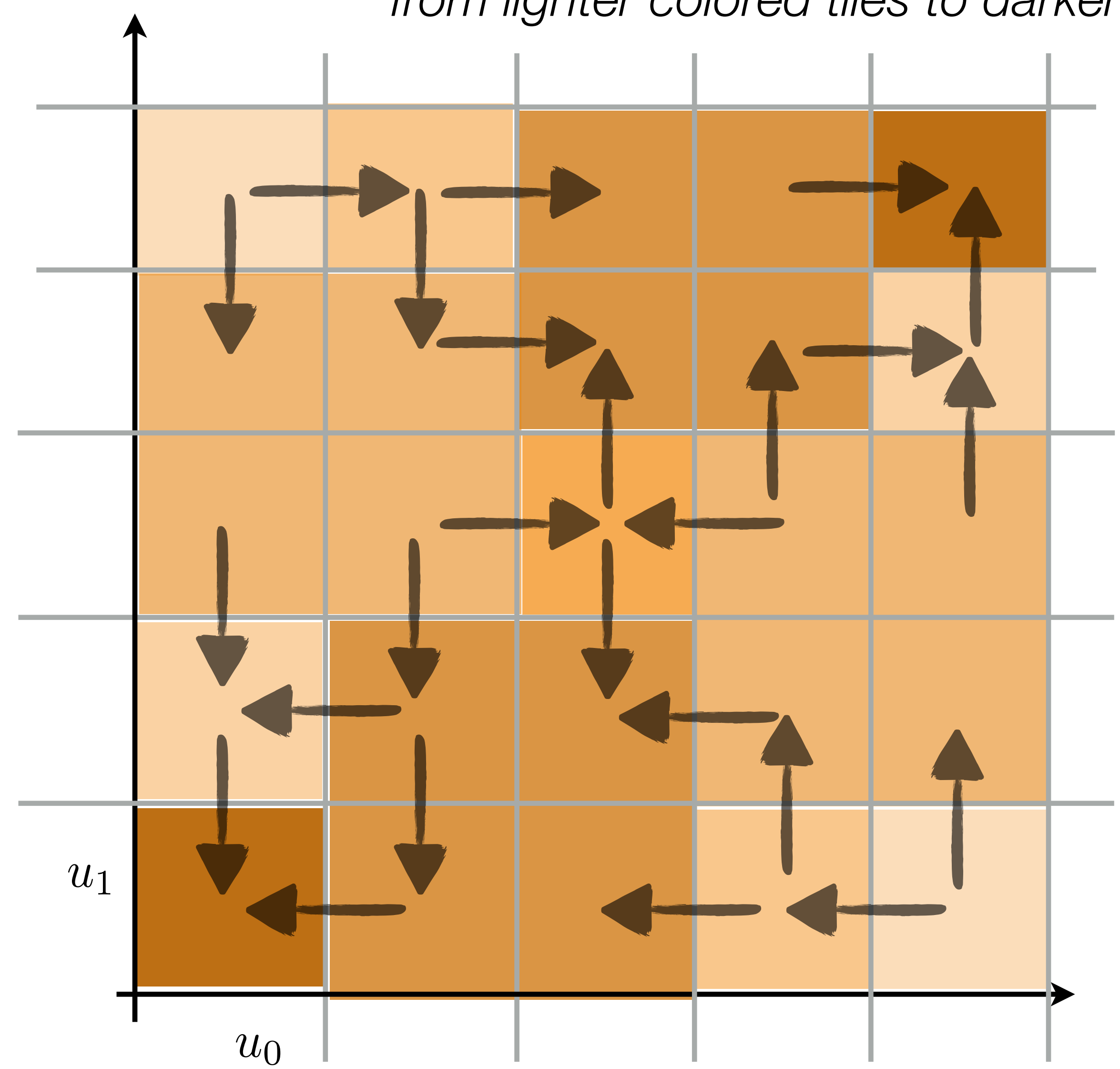
K. Mischaikow, Conley’s Connection Matrix (1987)

Morse theory on braids

van den Berg, Ghrist, van der Vorst, Inventiones Math. 2003



Solutions flow across boundary edges from lighter colored tiles to darker



Braided equilibrium solutions to parabolic PDE with periodic boundary conditions → Lattice filtered cubical complex in $\mathbb{R}^2$

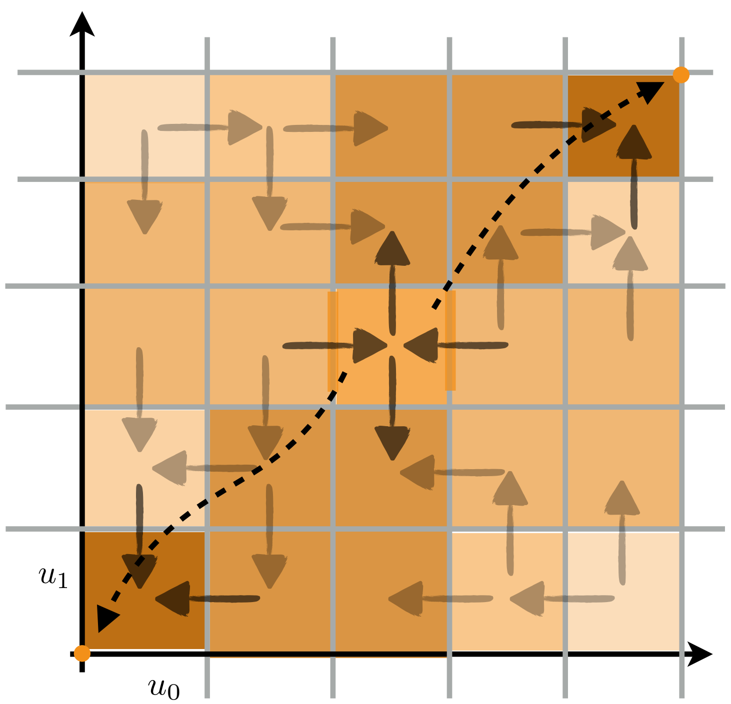
dynamics → *topological data*

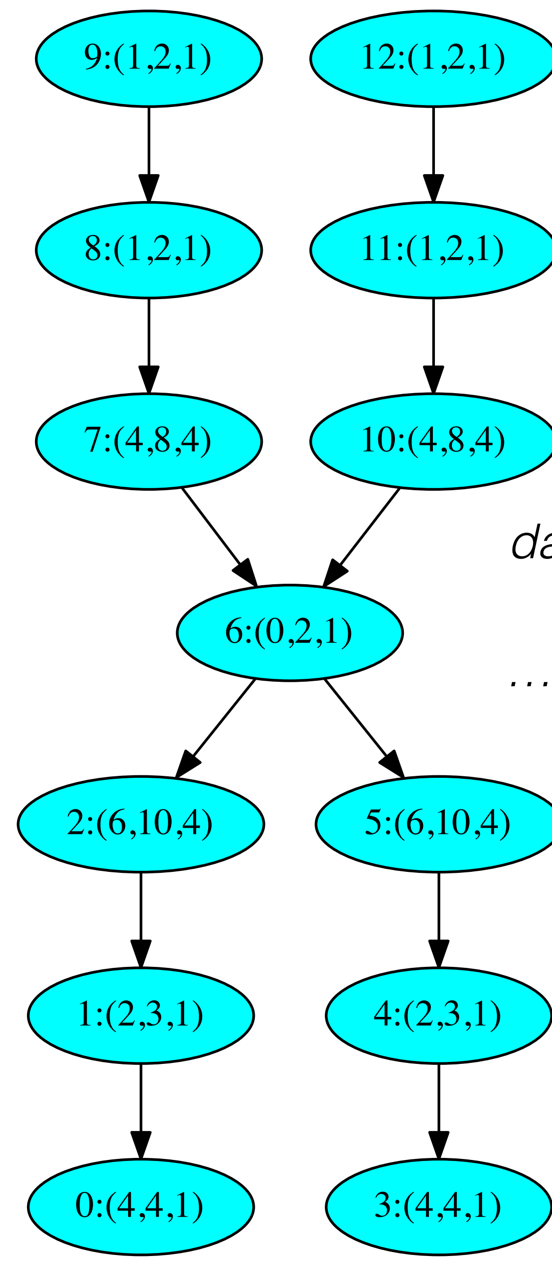
Fact: Nontrivial Conley indices imply existence of solutions to PDE

Fact: Nonzero entry in connection matrix between adjacent elements proves existence of connecting orbit

implementation I

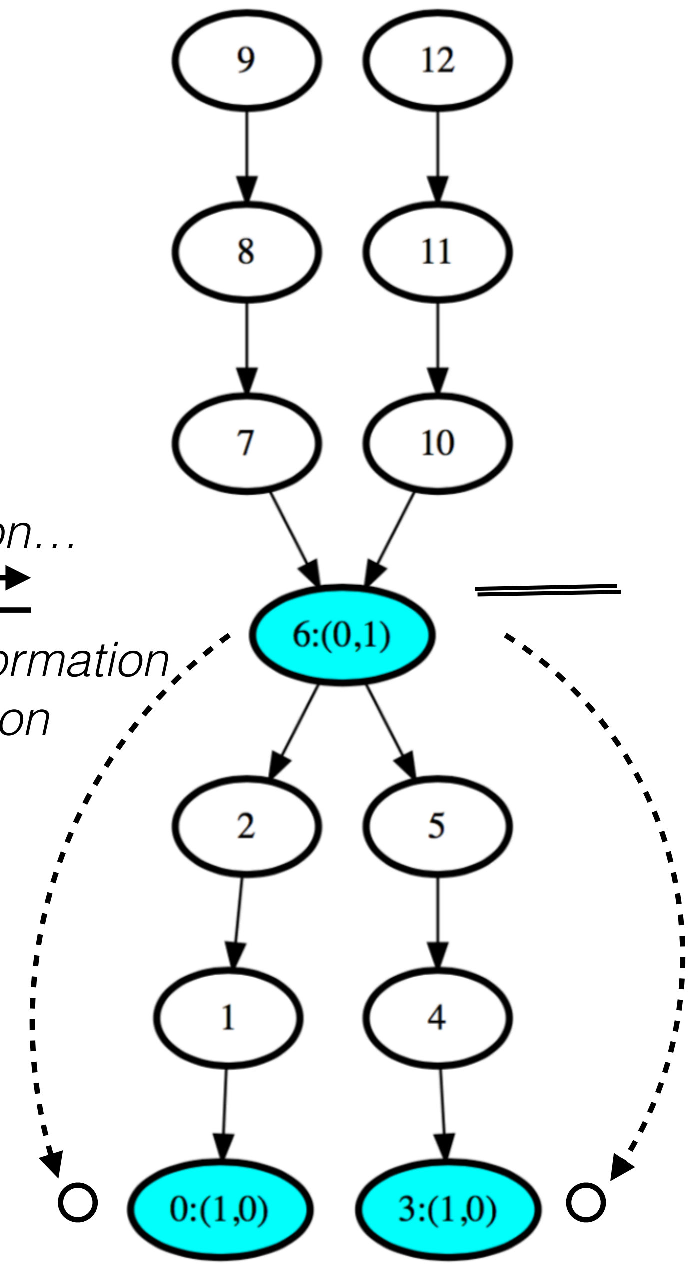
$$(X, \leq) \xrightarrow{\ \nu\ } (J(L), \leq)$$





data reduction…

…without information reduction



filtered reduction

reduced complex $M_\bullet$

$$\Delta^M = \begin{array}{cc|c|c|c} & & 0 & 3 & 6 \\ & & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 \\ \hline 3 & 0 & 0 & 0 & 1 \\ \hline 6 & 1 & 0 & 0 & 0 \end{array}$$

node index

cell dim.

data format at a node:

index of J(L) *: count of cells in the fiber for each dim.*

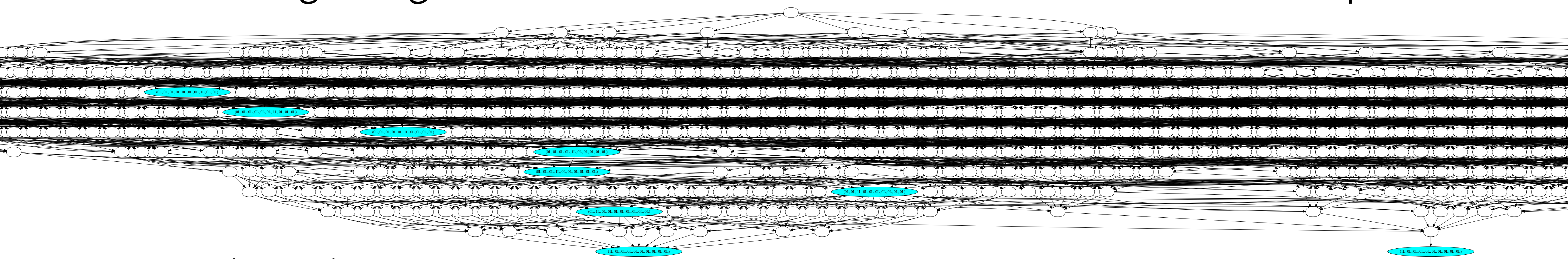
white nodes are trivial indices (no cells)

everything but nodes 0,3,6 have trivial Conley index

implementation II

data can get big

filtered cubical complex in $\mathbb{R}^9$

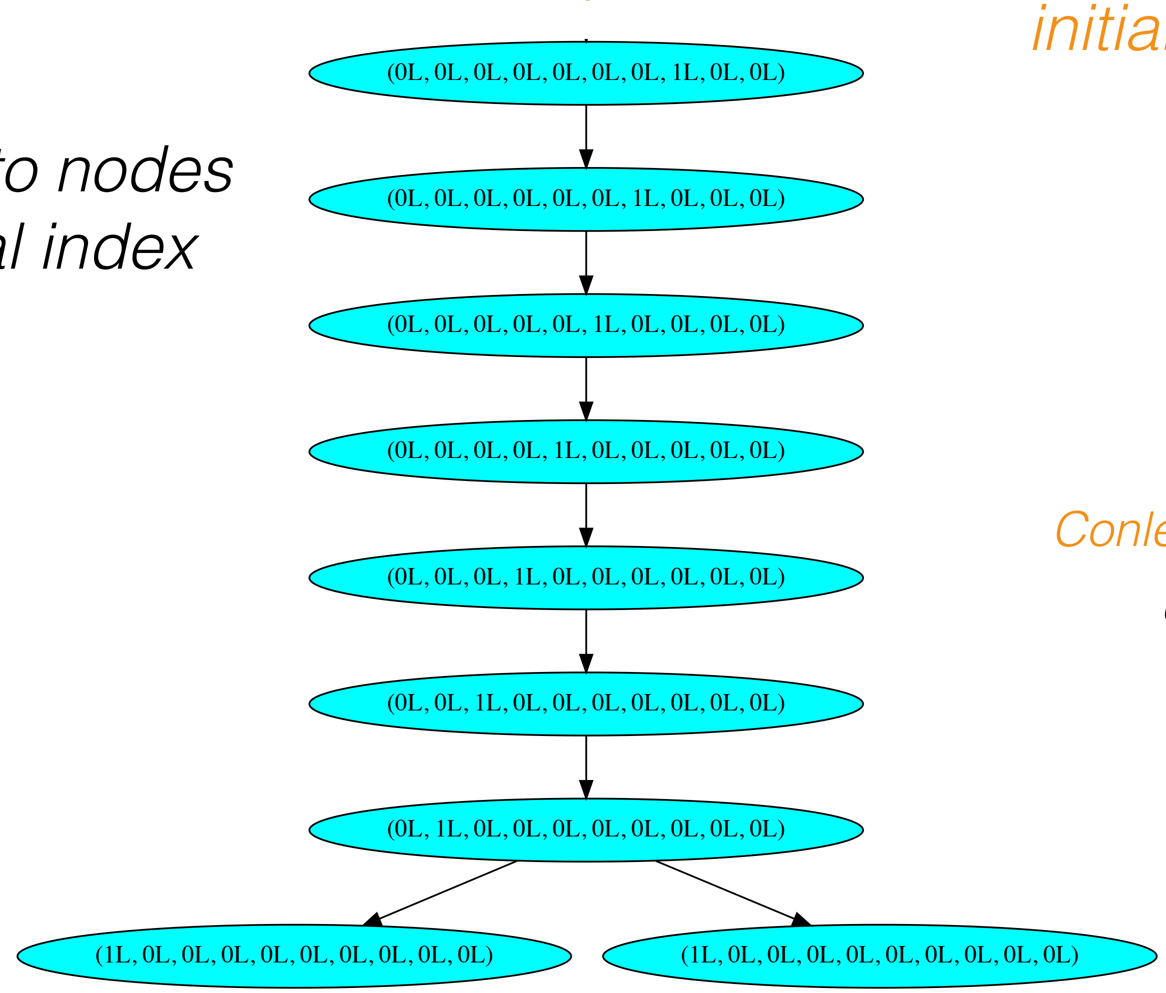


connection matrix

initial filtered cubical complex

10^9 cells $|\mathsf{J}(\mathsf{L})| \approx 900$

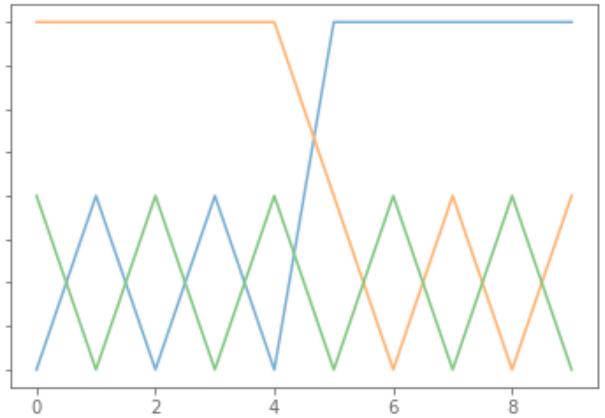
restrict poset to nodes with nontrivial index



Conley-Morse Graph

organizes global dynamics

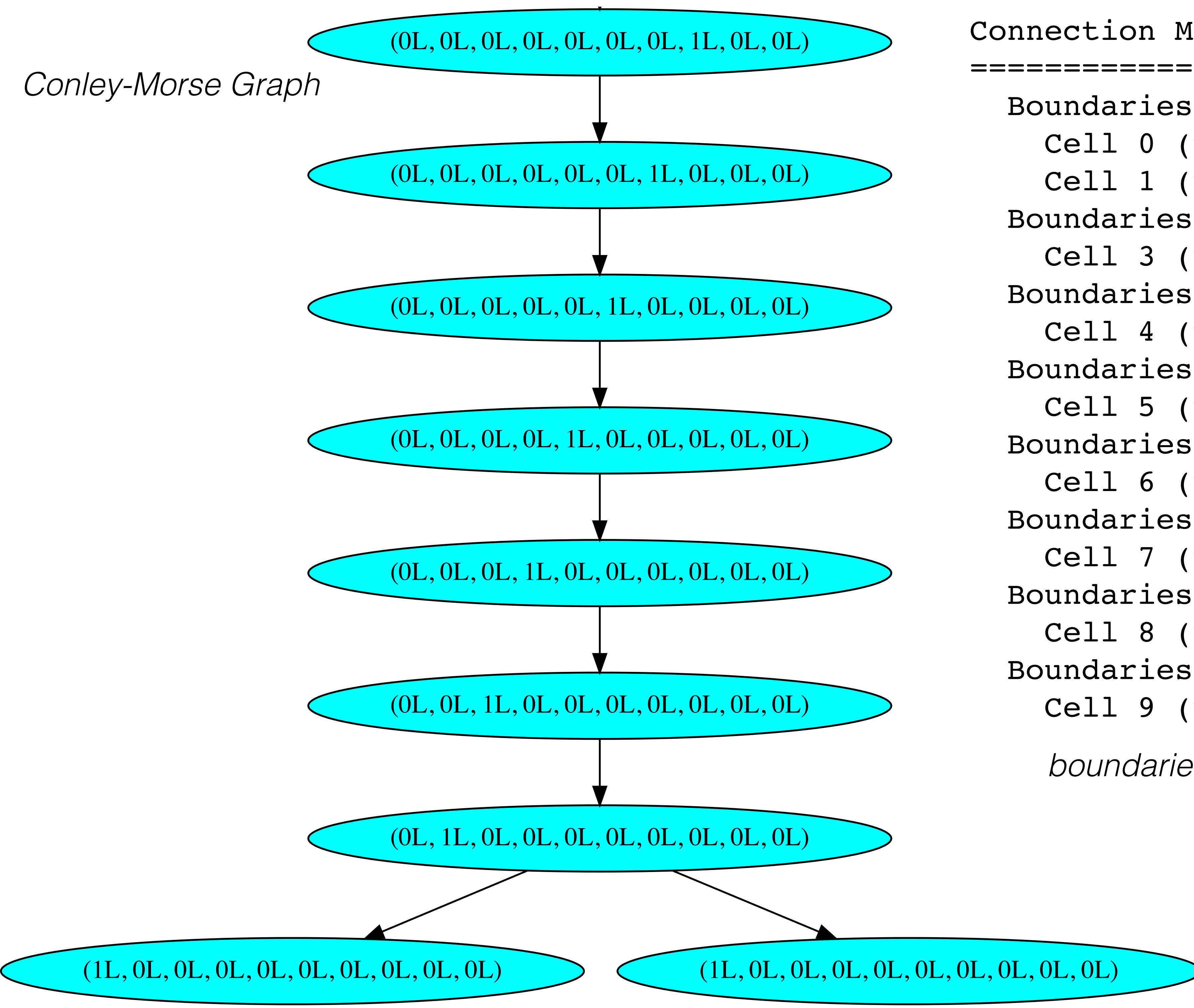
Conley index for each node



implementation III

order data

Conley-Morse Graph



chain data

9 cells

```
Connection Matrix Data
======================
  Boundaries of 0-cells (by cell index):
    Cell 0 (valuation 0) : set([])
    Cell 1 (valuation 4) : set([])
  Boundaries of 1-cells (by cell index):
    Cell 3 (valuation 12) : set([0L, 1L])
  Boundaries of 2-cells (by cell index):
    Cell 4 (valuation 33) : set([])
  Boundaries of 3-cells (by cell index):
    Cell 5 (valuation 67) : set([4L])
  Boundaries of 4-cells (by cell index):
    Cell 6 (valuation 111) : set([])
  Boundaries of 5-cells (by cell index):
    Cell 7 (valuation 160) : set([6L])
  Boundaries of 6-cells (by cell index):
    Cell 8 (valuation 209) : set([])
  Boundaries of 7-cells (by cell index):
    Cell 9 (valuation 688) : set([8L])
```

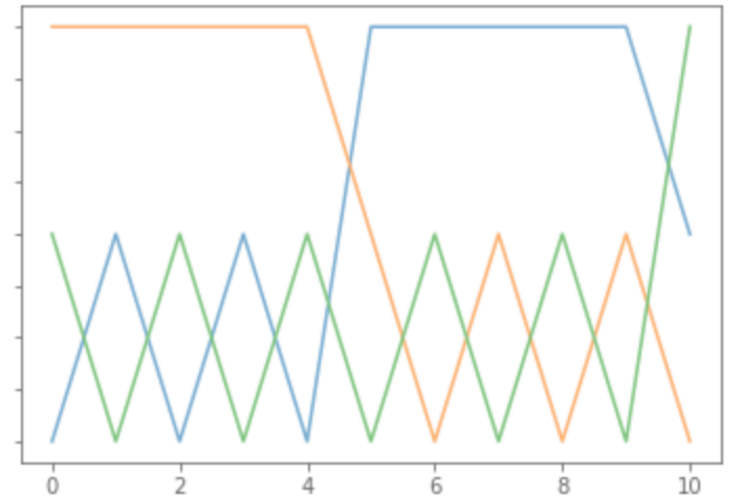
boundaries can be queried from the data structure

chain-level data reduction

10^9 cells $\rightleftarrows$ 9 cells

without loss of homological information

implementation IV



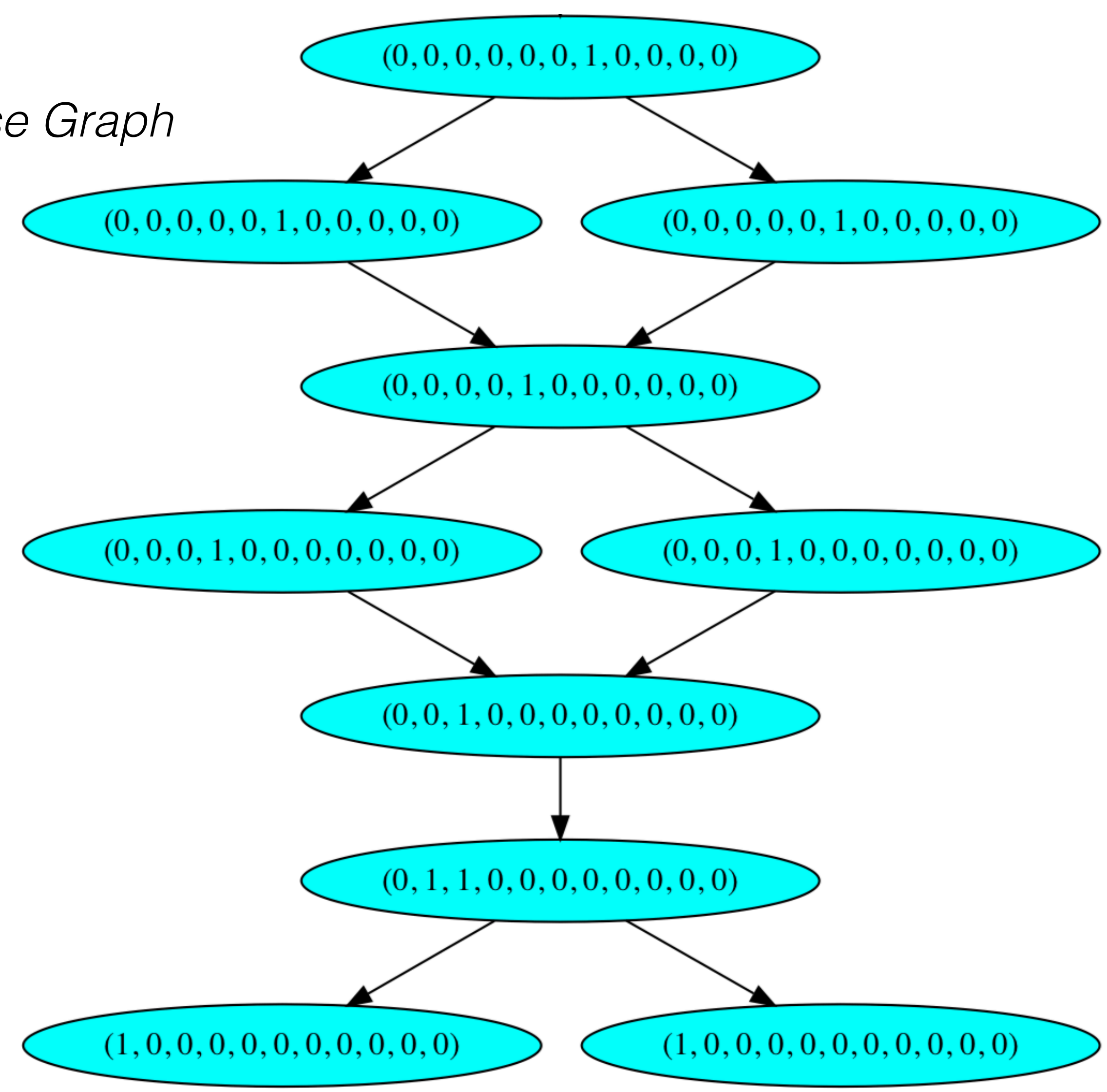
data can get even bigger

initial filtered cubical complex $\mathbb{R}^{10}$

10^{10} cells $|\mathsf{J}(\mathsf{L})| \approx 1775$

order data

Conley-Morse Graph



chain data

```
Connection Matrix Data
======================
Boundaries of 0-cells:
 0 : set()
 1 : set()
Boundaries of 1-cells:
 2 : {0, 1}
Boundaries of 2-cells:
 3 : set()
 4 : set()
Boundaries of 3-cells:
 5 : {3, 4}
 6 : {3}
Boundaries of 4-cells:
 7 : set()
Boundaries of 5-cells:
 8 : {7}
 9 : {7}
Boundaries of 6-cells:
 10 : {8, 9}
```

chain-level data reduction

10^{10} cells $\rightleftarrows$ 11 cells thank you for your attention

Collaborators:
S. Harker
K. Mischaikow
R. van der Vorst

NSF