Computational Connection Matrix Theory

…toward new tools in applied topology

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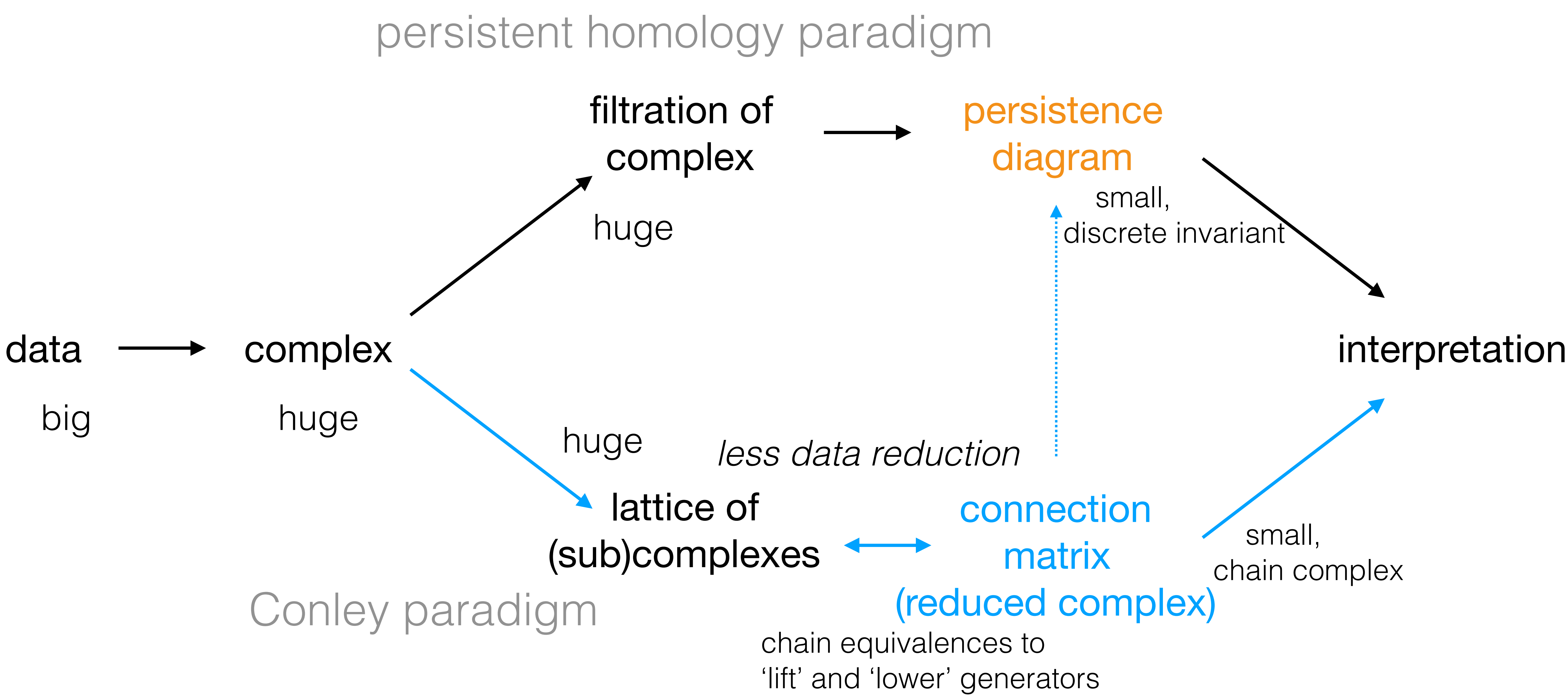
Workshop on Applied Topology, 2019

workflows



persistent homology paradigm
filtration of complex
persistence diagram
huge
small, discrete invariant
data
big
complex
huge
interpretation
huge
less data reduction
lattice of (sub)complexes
connection matrix (reduced complex)
small, chain complex
Conley paradigm
chain equivalences to 'lift' and 'lower' generators

Data Structures

graded complexes

X cell complex
e.g. Lefschetz, CW

(X, $\leq$) face poset

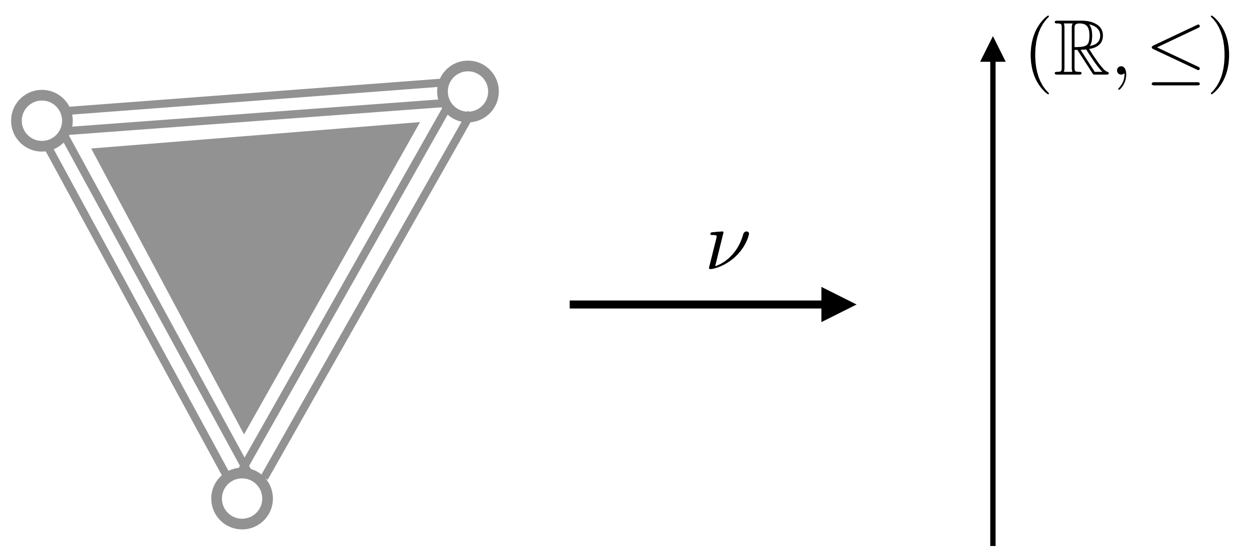
$\mathbb{R}$ poset

an order preserving map (X, $\leq$) $\xrightarrow{\nu}$ $(\mathbb{R}, \leq)$

filters X via pre-images of downsets

$\nu^{-1}(-\infty, a]$ is a subcomplex of X

the collection $\{\nu^{-1}(-\infty, a]\}_{a \in \mathbb{R}}$ is a filtration



$U \subseteq \mathbb{R}$ is a down-set if the following holds: $x \in U$ and $y \leq x$ implies $y \in U$

X cell complex
e.g. Lefschetz, CW

(X, $\leq$) face poset

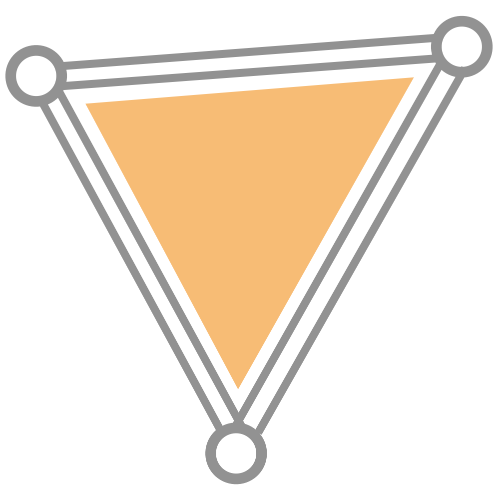
P finite poset

an order preserving map $(\mathsf{X}, \leq) \xrightarrow{\nu} (\mathsf{P}, \leq)$

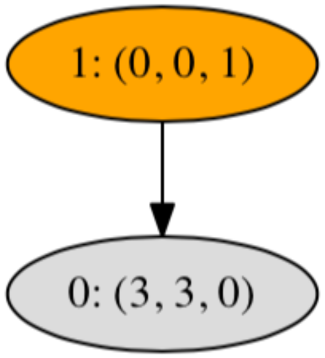
filters X via pre-images of downsets

$\{\nu^{-1}(U)\}_{U \in \mathsf{O}(\mathsf{P})}$ is a lattice of subcomplexes

one-critical multifiltration if $\mathsf{P} \subset \mathbb{R}^n$



$\xrightarrow{\nu}$



count of cells in the fiber for each dim.

the lattice of down sets O(P) is

$\mathsf{O}(\mathsf{P}) := \{U \subset \mathsf{P} : U \text{ is a downset}\}$ $\quad \wedge := \cap \quad \vee := \cup$ *Birkhoff's Theorem*

X cell complex
(Lefschetz, CW)

(X, $\leq$) face poset

P finite poset

Definition (P-graded cell complex)

X, P, and a poset morphism ν from X to P

$$(\mathsf{X}, \leq) \xrightarrow{\nu} (\mathsf{P}, \leq)$$

a graded cell complex determines

a P-graded chain complex $(C(\mathsf{X}), \partial)$

$$C(\mathsf{X}) = \bigoplus_{p \in \mathsf{P}} C(\nu^{-1}(p))$$

boundary map is P-graded

$$\partial_{pq} \neq 0 \implies p \leq q$$

upper triangular wrt P

for a graded chain complex $C(\mathsf{X}) = \bigoplus_{p \in \mathsf{P}} C(\nu^{-1}(p))$

structure is determined by the fibers of ν

Definition (cyclic **P**-graded complex)

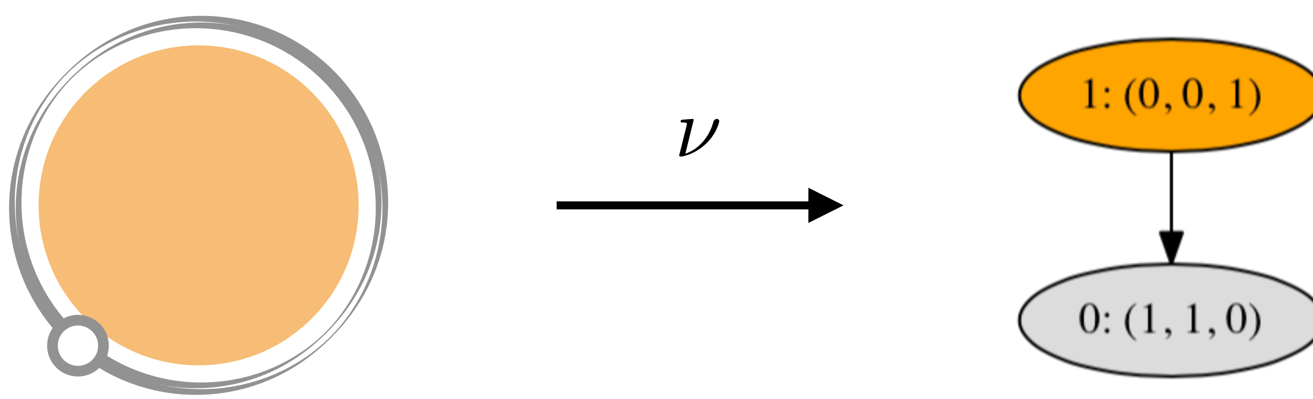
P-graded complex with cyclic fibers

$\partial_{pp} = 0$ for ***p*** in **P**

'small' objects

i.e. ∂ is strictly upper triangular wrt **P**

Betti numbers of fiber
(Conley index)



goal: replace graded complex with equivalent cyclic graded complex

Categories

goal: homologically-faithful data compression category **GCh(P)** of **P**-graded chain complexes

morphisms: **P**-graded chain maps

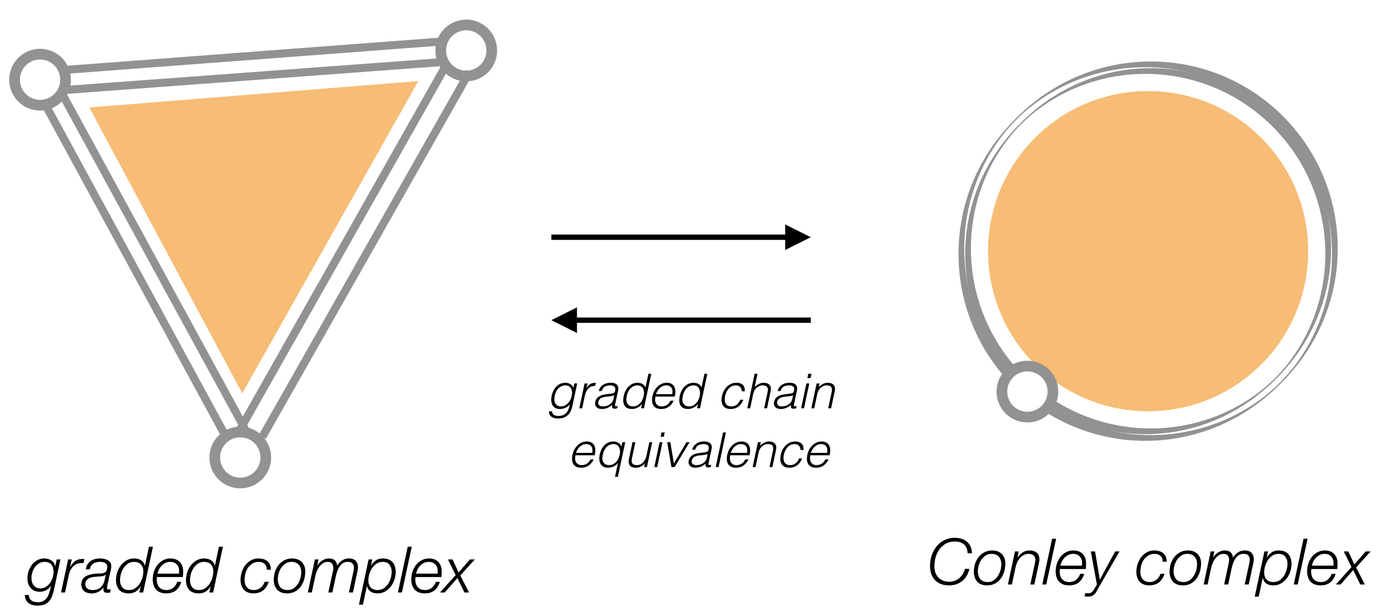
homotopy category **GK(P)** of **P**-graded chain complexes

localize about **P**-graded chain equivalences

interpretation of connection matrix for data analysis:

a *Conley complex* is a cyclic representative of isomorphism class in **GK(P)**

the boundary operator of a Conley complex is a *connection matrix*



moral: homotopy categories for chain-level data compression without loss of homological information

subcategory **$GK_0(P)$** of cyclic **P**-graded complexes

$$GK_0(P) \xrightarrow{\mathfrak{I}} GK(P)$$

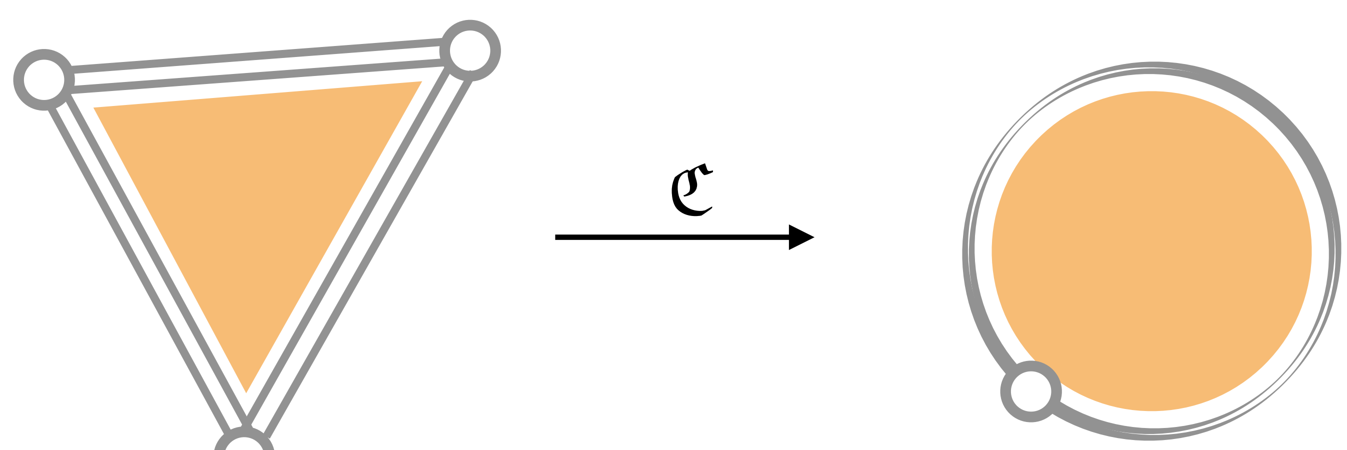
Theorem: over fields, the inclusion functor $\mathfrak{I}$ is full, faithful and essentially surjective (categorical equivalence)

thus there exists an inverse functor $\mathfrak{C}$ called a Conley functor

$$GK_0(P) \underset{\mathfrak{C}}{\overset{\mathfrak{I}}{\rightleftarrows}} GK(P)$$

taking a graded chain complex to a Conley complex

under the hood: discrete morse theory

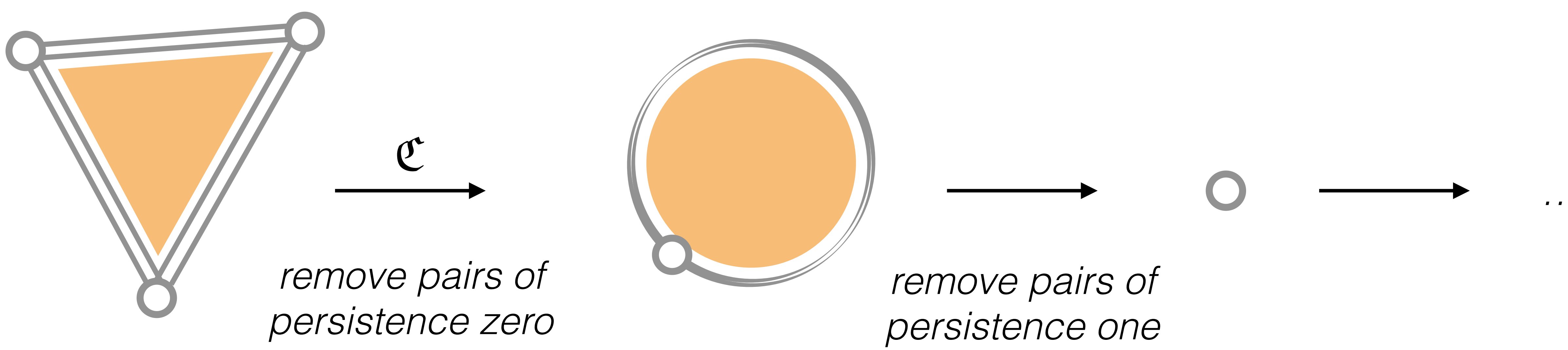


graded complex *Conley complex*

Harker + Mischaikow + S.

moral: homotopy categories for chain-level data compression without loss of homological information

Theorem: For any graded complex the persistent homology groups of $C_\bullet(\mathsf{P})$ and $\mathfrak{C}(C_\bullet(\mathsf{P}))$ are isomorphic



Remark: When **P** is a total order computing a Conley complex is the beginning of a spectral sequence-type algorithm (Edelsbrunner & Harer, Bauer et al, ...)

Harker + Mischaikow + S.

computational Conley homology

applications + implementation

https://github.com/shaunharker/pyCHomP

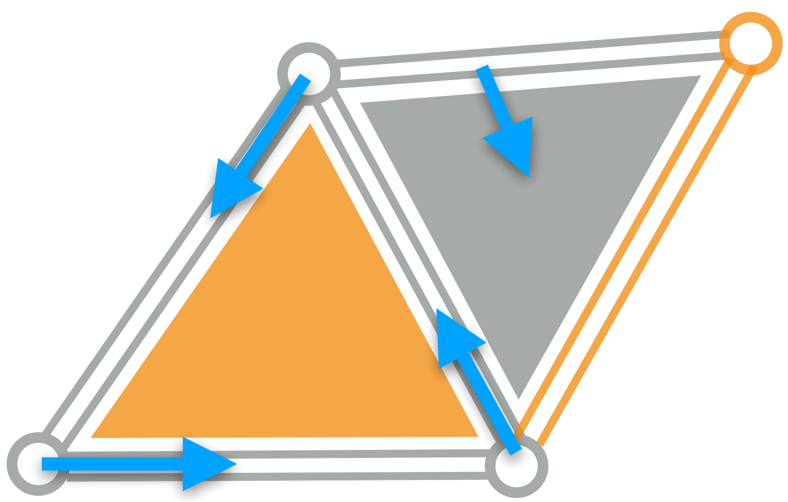
application i:

discrete flows

discrete flows

the simplest discrete flow is a *combinatorial vector field* on simplicial complex $\mathcal{K}$ (Forman)

a combinatorial vector field is a partial matching $\{c_i\} \sqcup \{y_i < x_i\}$
two cells are matched only if one is a codimension-1 face of the other
(no acyclicity requirement)



arrows give the matching

cells are critical if they are not matched

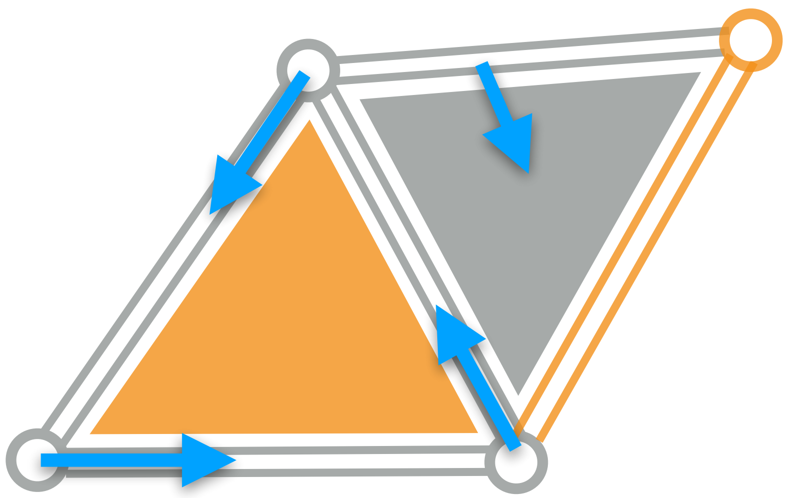
face poset and matching give a directed graph $\mathcal{F}$ on complex $\mathcal{K}$
$x_i \to y_i$ if $y_i < x_i$ are matched or y_i is a codimension-1 face of x_i

$\mathcal{F}$ partitions $\mathcal{K}$ into poset of strongly connected components $\mathsf{SC}(\mathcal{F})$

$a \in \mathsf{SC}(\mathcal{F})$ iff it is a (maximal) set of vertices such that
for any $\xi, \xi' \in a$ there are paths $\xi \to \xi'$ and $\xi' \to \xi$ in $\mathcal{F}$

discrete flows ii

complex partitions into poset of strongly connected components

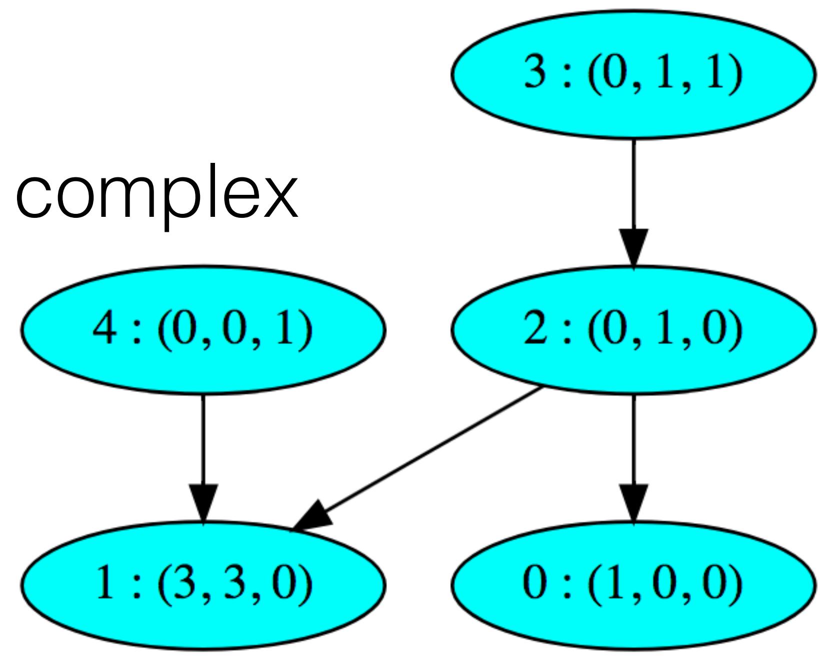


$\rightleftarrows$

graded homotopy equivalence

poset organizes the global dynamics

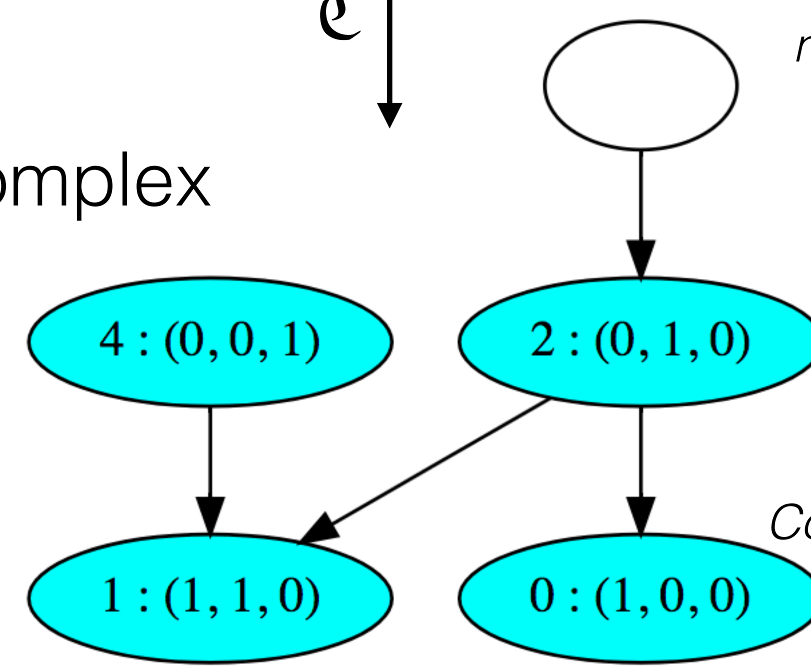
graded cell complex



$\Big\downarrow \mathfrak{C}$

white nodes have no cells (trivial index)

Conley complex



Conley index quantifies local dynamics

Remark: computational connection matrix theory generalizes to multi-vectors of Mrozek

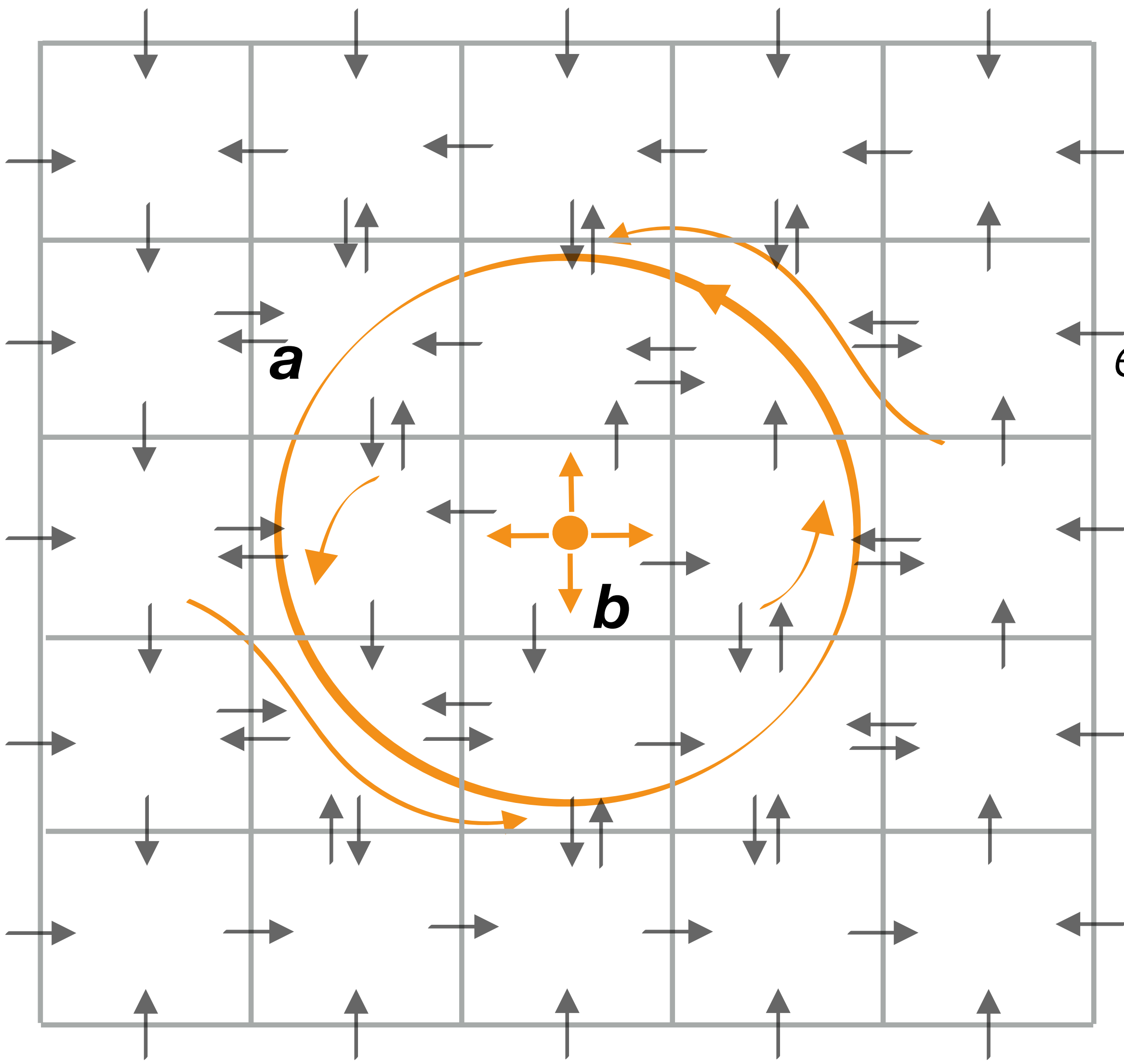
application ii:

transversality

topological space is approximated with a cell complex $\mathcal{X}$

continuous dynamics are approximated

with directed graph $\mathcal{F}$ on top-dimensional cells $\mathcal{X}_n$



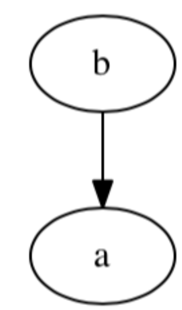
grade $\mathcal{X}$ via:

poset $\mathsf{SC}(\mathcal{F})$ strongly connected components of $\mathcal{F}$

every vertex belongs to strongly connected component

map from top cells to strongly connected components

$$(\mathcal{X}_n, \leq) \longrightarrow \mathsf{SC}(\mathcal{F})$$



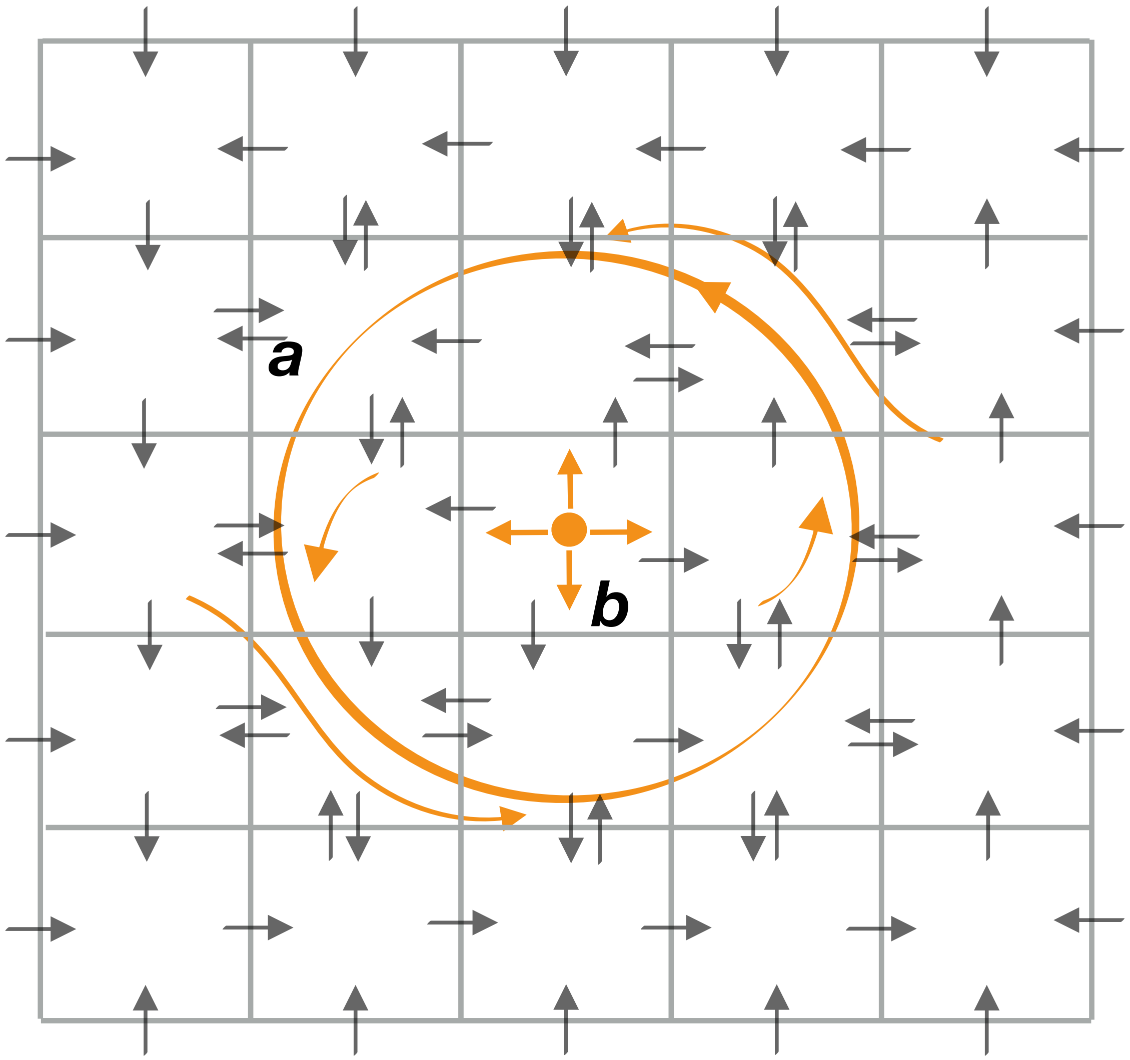
need to extend to graded cell complex

$$\begin{tikzcd} (\mathcal{X}_n, \leq) \arrow[r] \arrow[d] & (\mathcal{X}, \leq) \arrow[dl, dashed, "\nu"] \\ \mathsf{SC}(\mathcal{F}) & \end{tikzcd}$$

graded cell complex

φ is the flow generated by $\dot{x} = f(x)$

transversality models

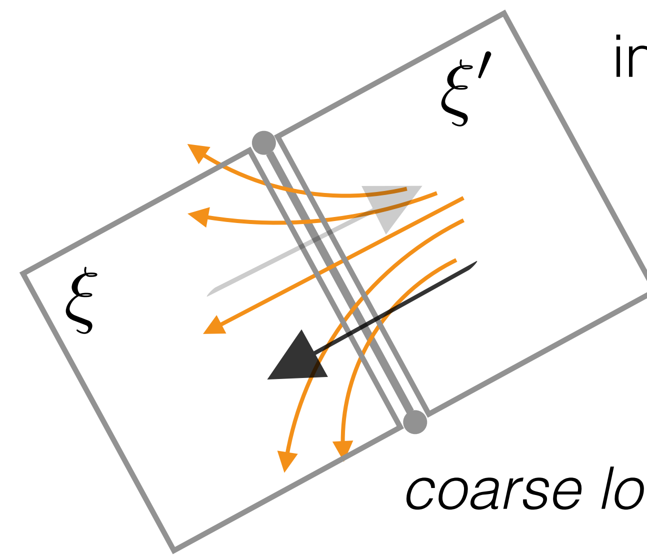


transversality model

if $\xi \not\to \xi'$ between adjacent top cells

then the flow φ is transverse to $\xi \cap \xi'$

in the direction $\xi' \to \xi$



coarse lower bound on dynamics

Theorem: if the graph is a transversality model then there is an extension ν

$$
\begin{tikzcd}
(\mathcal{X}_n, \leq) \arrow[r] \arrow[d] & (\mathcal{X}, \leq) \arrow[dl, dashed, "\nu"] \\
\mathsf{SC}(\mathcal{F}) &
\end{tikzcd}
$$

graded cell complex

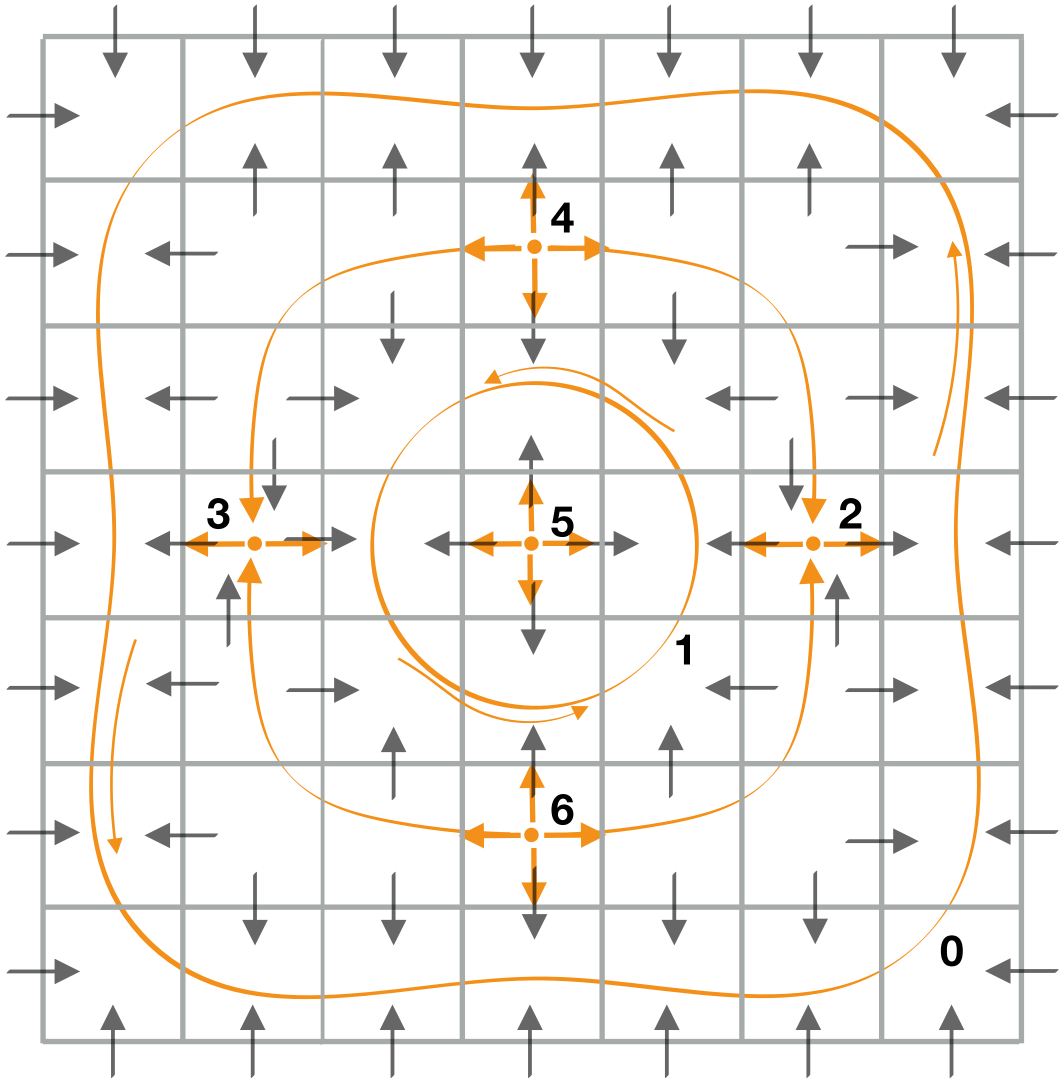
1. $\mathsf{A} = \{\nu^{-1}(a)\}_{a \in \mathsf{O}(\mathsf{SC}(\mathcal{F}))}$ is a lattice of attracting blocks for φ

2. $\mathfrak{C}(\mathcal{X}, \nu)$ is a Conley complex for continuous dynamics φ

Remark: Computations + theorems are valid for any differential equation which is transverse to top cell boundaries in direction indicated

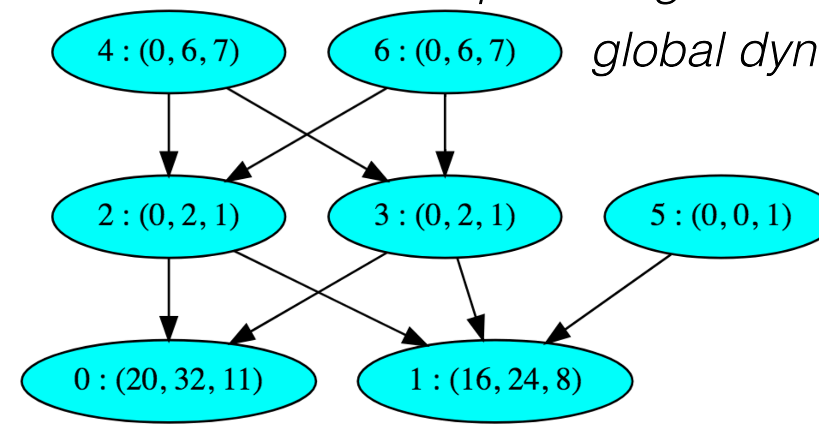
Harker + Mischaikow + S. + Vandervorst

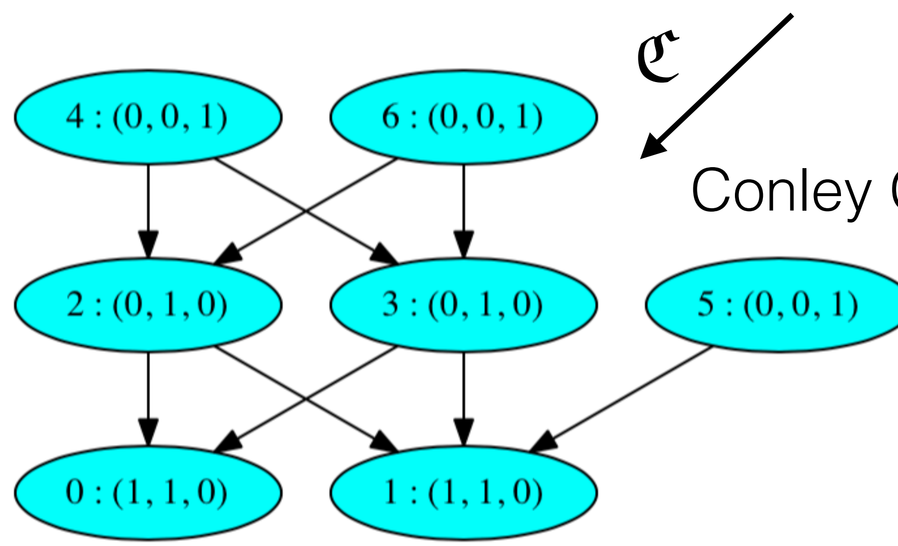
transversality models ii



graded cell complex

poset organizes the global dynamics





Conley Complex

connection matrix is represented
with respect to a basis

different bases give different
qualitative descriptions of dynamics

in this example: four different bases

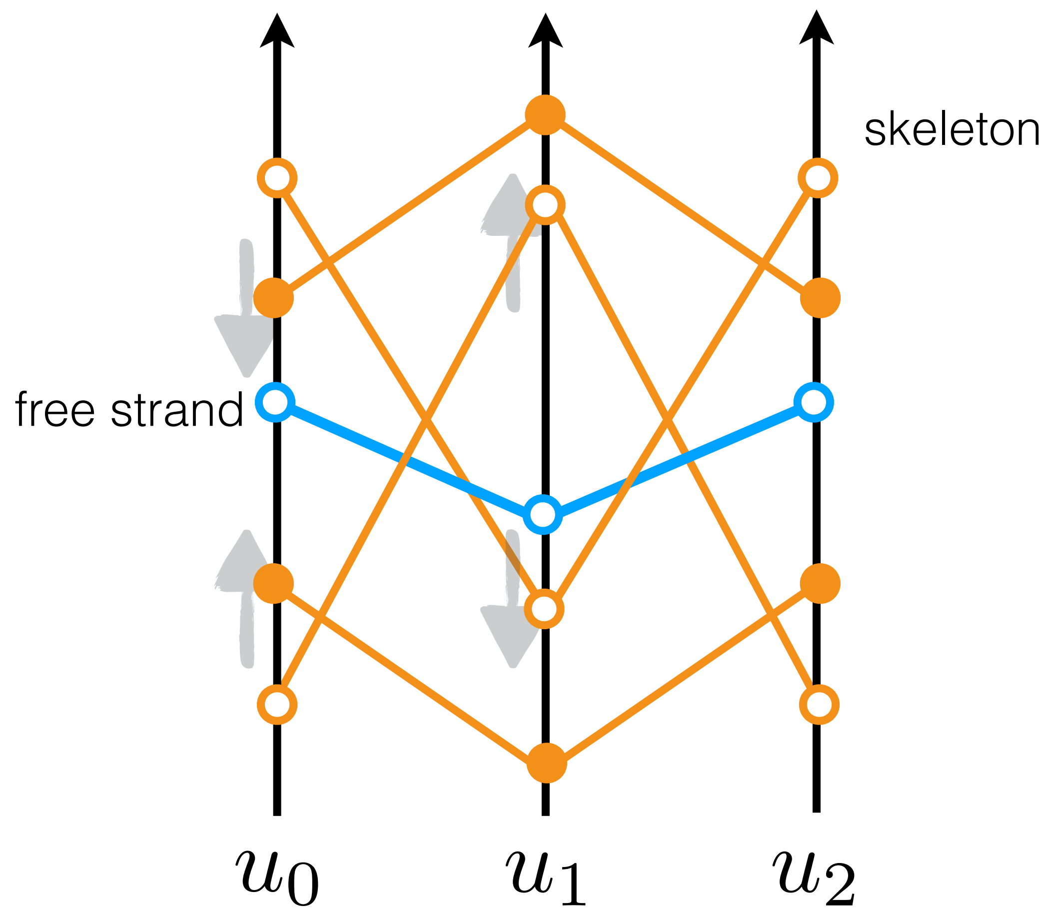
Connection Matrix Data
==================
Boundaries of 0-cells (by cell index):
Cell 0 (valuation 1) : {}
Cell 1 (valuation 0) : {}
Boundaries of 1-cells (by cell index):
Cell 2 (valuation 2) : {0, 1}
Cell 3 (valuation 3) : {0, 1}
Cell 4 (valuation 0) : {}
Cell 5 (valuation 1) : {}
Boundaries of 2-cells (by cell index):
Cell 6 (valuation 6) : {2, 3, 4, 5}
Cell 7 (valuation 5) : {5}
Cell 8 (valuation 4) : {2, 3}

application iii:

Morse theory on braids

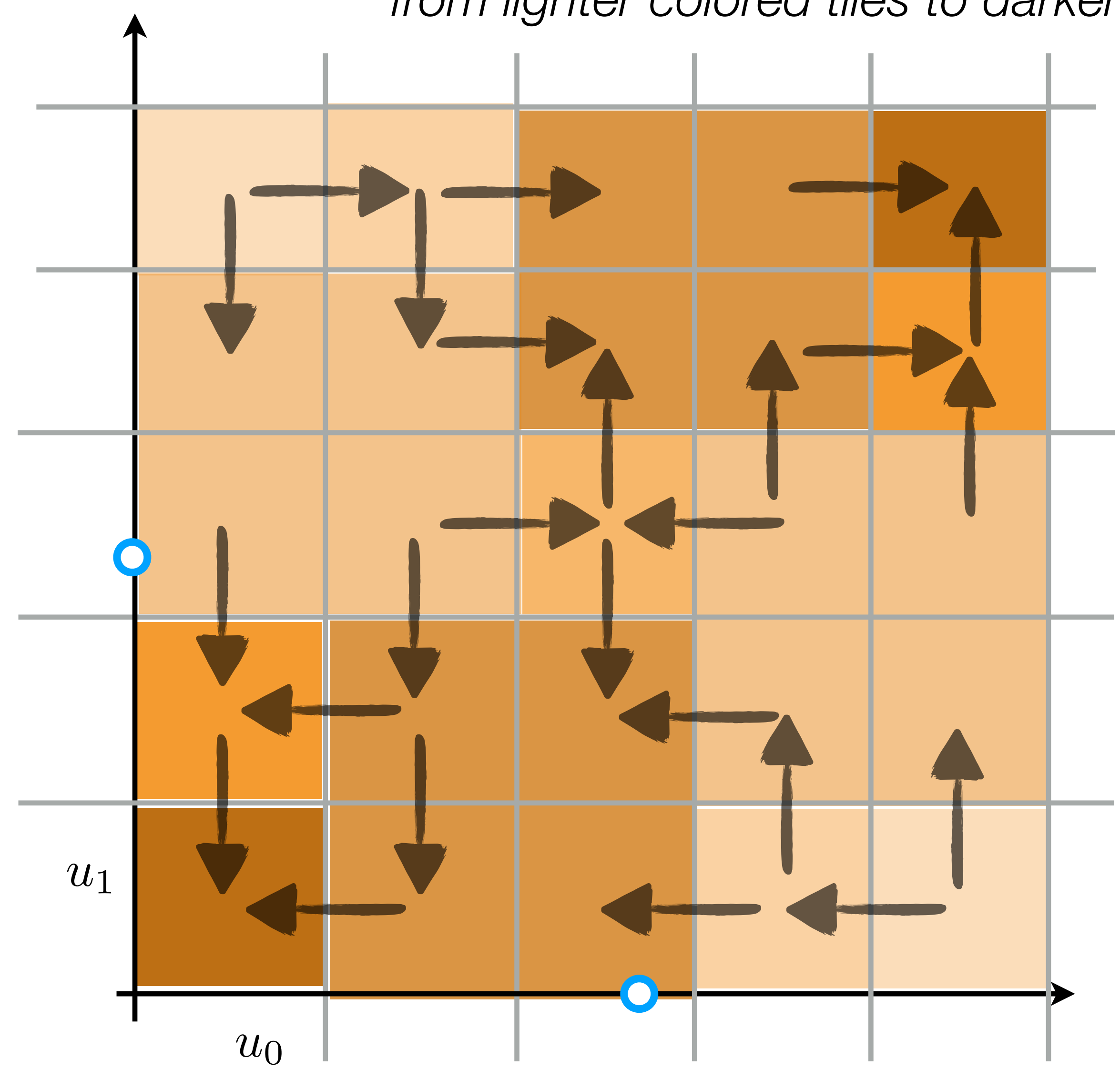
Morse theory on braids

van den Berg, Ghrist, van der Vorst,
Inventiones Math. 2003



Braided equilibrium solutions to parabolic PDE with periodic boundary conditions →

Solutions flow across boundary edges from lighter colored tiles to darker



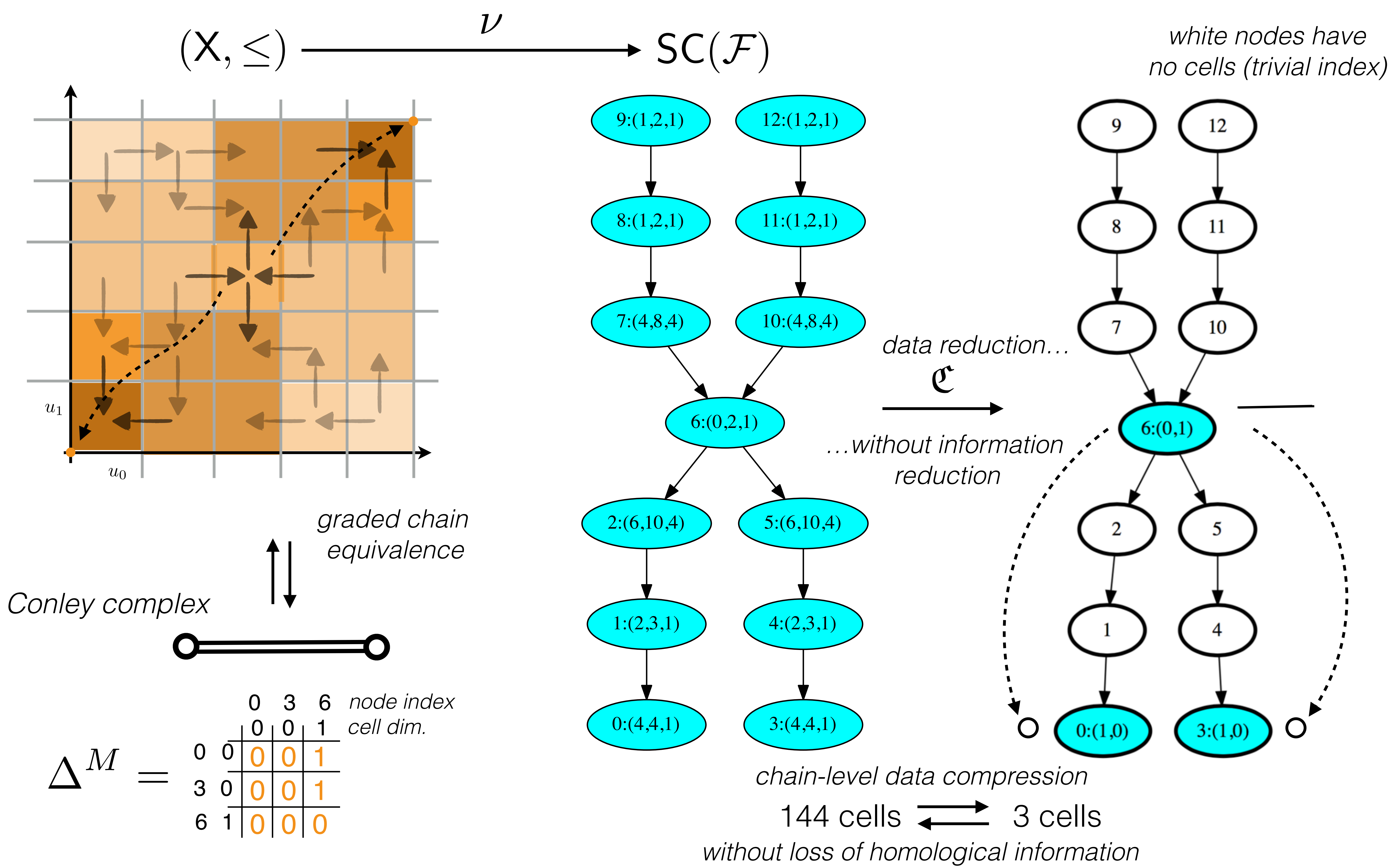
transversality model in $\mathbb{R}^2$

graded cubical complex in $\mathbb{R}^2$

Fact: Nontrivial Conley indices imply existence of solutions to PDE

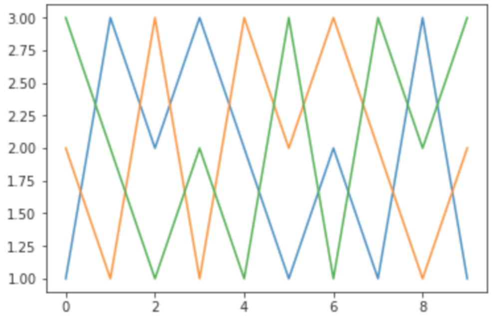
Fact: Nonzero entry in connection matrix between adjacent elements proves existence of connecting orbit

braids i



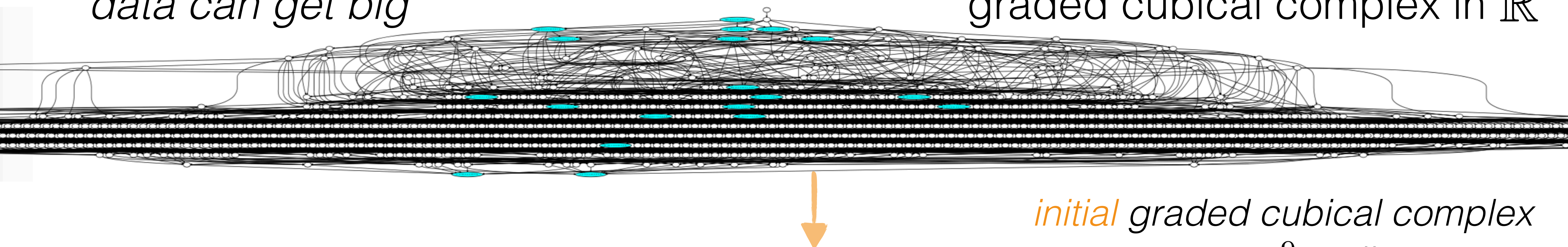
(X, ≤)
ν
SC(𝓕)
white nodes have no cells (trivial index)
u_1
u_0
9:(1,2,1)
12:(1,2,1)
8:(1,2,1)
11:(1,2,1)
7:(4,8,4)
10:(4,8,4)
6:(0,2,1)
2:(6,10,4)
5:(6,10,4)
1:(2,3,1)
4:(2,3,1)
0:(4,4,1)
3:(4,4,1)
data reduction…
…without information reduction
9
12
8
11
7
10
6:(0,1)
2
5
1
4
0:(1,0)
3:(1,0)
graded chain equivalence
Conley complex
node index
cell dim.
$\Delta^M =$
chain-level data compression
144 cells
3 cells
without loss of homological information

braids ii



data can get big

graded cubical complex in $\mathbb{R}^9$

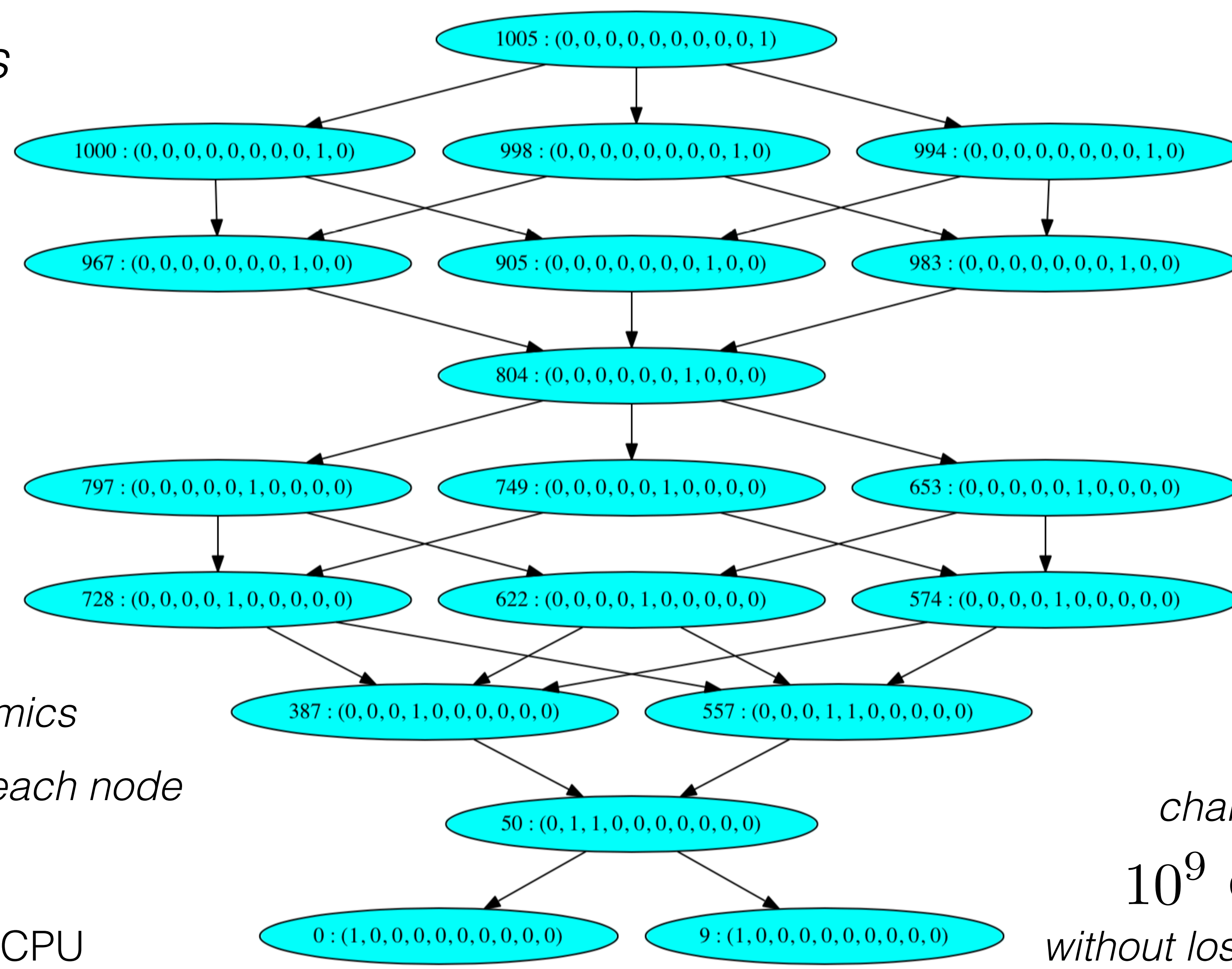


initial graded cubical complex

10^9 cells

$|\mathsf{SC}(\mathcal{F})| \approx 1000$

restrict poset to nodes with nontrivial index



Conley complex

21 cells

19 nodes

Conley-Morse Graph

organizes global dynamics

Conley index for each node

chain-level data compression

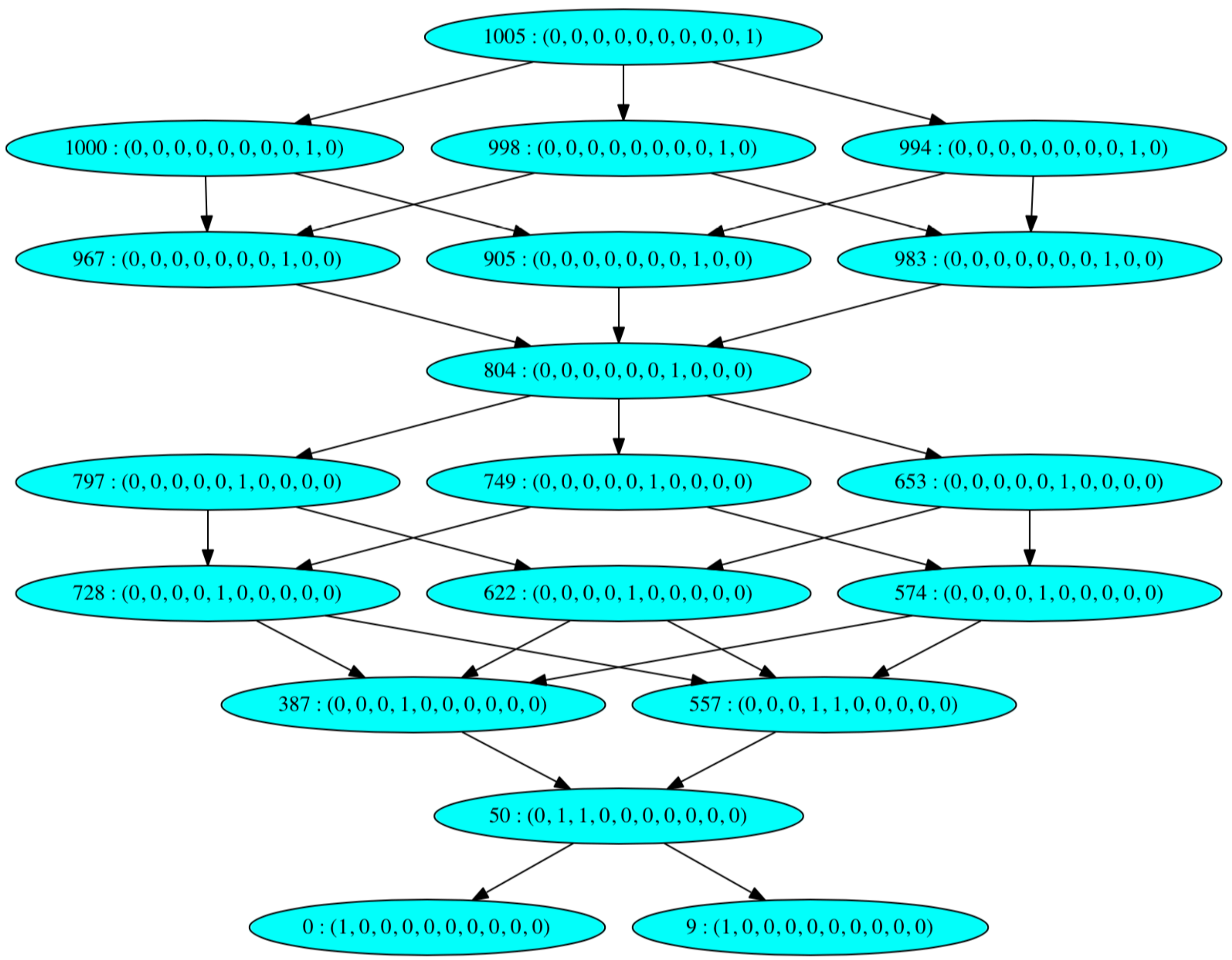
10^9 cells $\rightleftarrows$ 21 cells

without loss of homological information

~47 min on single 2.5 GHz CPU

braids iii

order data



chain data

```
Connection Matrix Data
======================
  Boundaries of 0-cells in Conley complex:
    0 : set()
    1 : set()
  Boundaries of 1-cells in Conley complex:
    2 : {0, 1}
  Boundaries of 2-cells in Conley complex:
    3 : set()
  Boundaries of 3-cells in Conley complex:
    4 : {3}
    5 : {3}
  Boundaries of 4-cells in Conley complex:
    6 : {4, 5}
    7 : {4, 5}
    8 : {4, 5}
    9 : set()
  Boundaries of 5-cells in Conley complex:
    10 : {8, 9, 6}
    11 : {8, 9, 7}
    12 : {9, 6, 7}
  Boundaries of 6-cells in Conley complex:
    13 : set()
  Boundaries of 7-cells in Conley complex:
    14 : {13}
    15 : {13}
    16 : {13}
  Boundaries of 8-cells in Conley complex:
    17 : {14, 15}
    18 : {16, 14}
    19 : {16, 15}
```

Conley-Morse Graph

organizes global dynamics

Conley index for each node

Conley Complex *connection matrix*

boundaries can be queried from the data structure

chain maps to move between data and invariant thank you for your attention

Collaborators:
S. Harker
K. Mischaikow
R. van der Vorst

Harker, S., Mischaikow, K. and Spendlove, K., 2018.
A Computational Framework for the Connection Matrix Theory.
arXiv preprint arXiv:1810.04552.