Morse, Conley, and Computation

…toward a computational homological theory of dynamics

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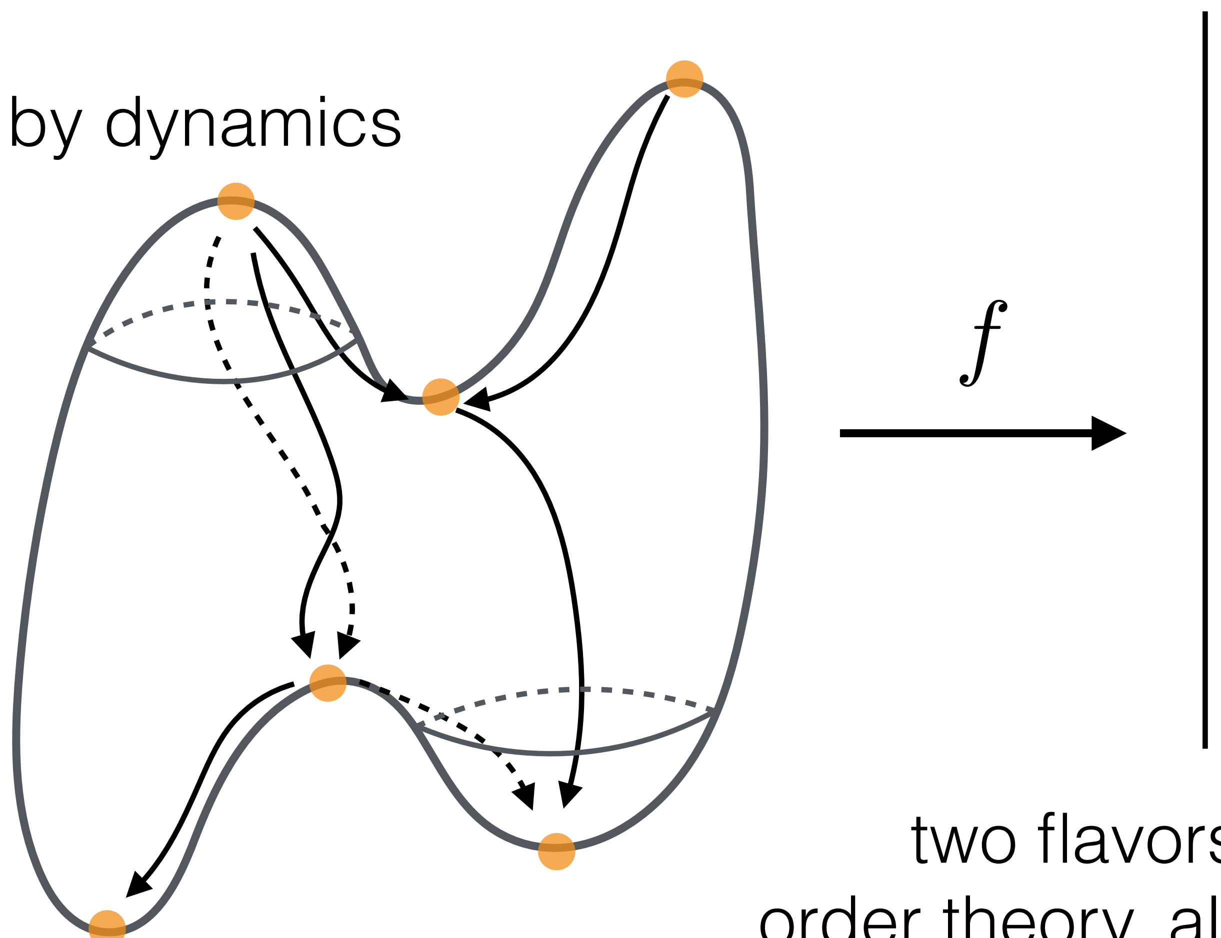
dynamical musings

- a dynamical system engenders topological data
- local data (e.g. equilibria) and global data (attractors)
- topological data are ordered and measured with algebra

data are *ordered* by dynamics

$$\dot{x} = -\nabla f$$



data have *algebraic invariants* (homology)

two flavors of algebra:
order theory, algebraic topology

Conley-Morse Theory

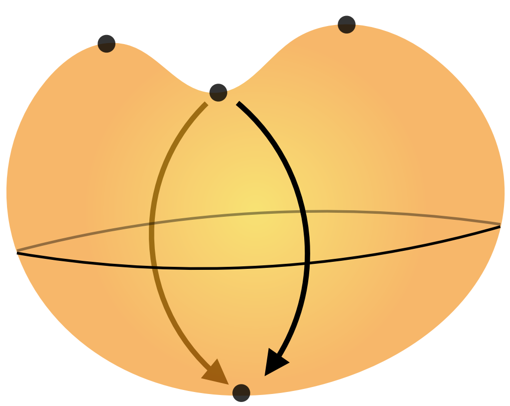
‘…if such rough equations are to be of use it is necessary to study them in rough terms.’
C. Conley, CBMS Monograph (1978)

first, the model of a (Morse-type) gradient system

Morse theory

list of ingredients

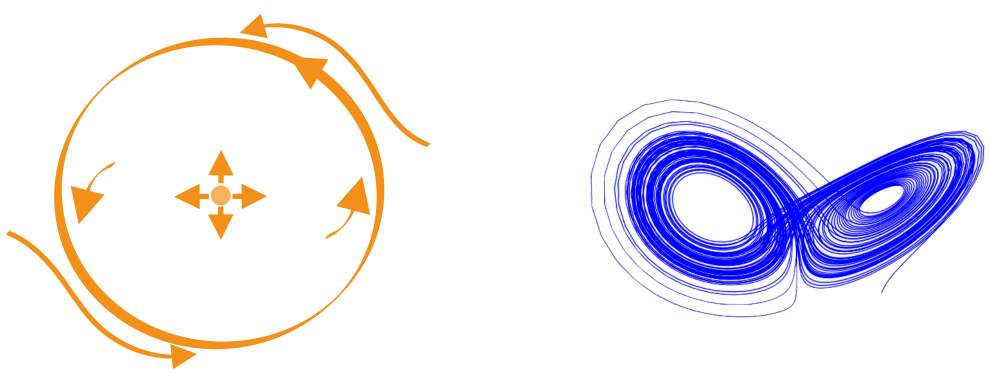
- Morse index
- gradient structure (height function)
- Morse homology



Conley theory

list of ingredients

- Conley index
- lattice (of attractors)
- connection matrix



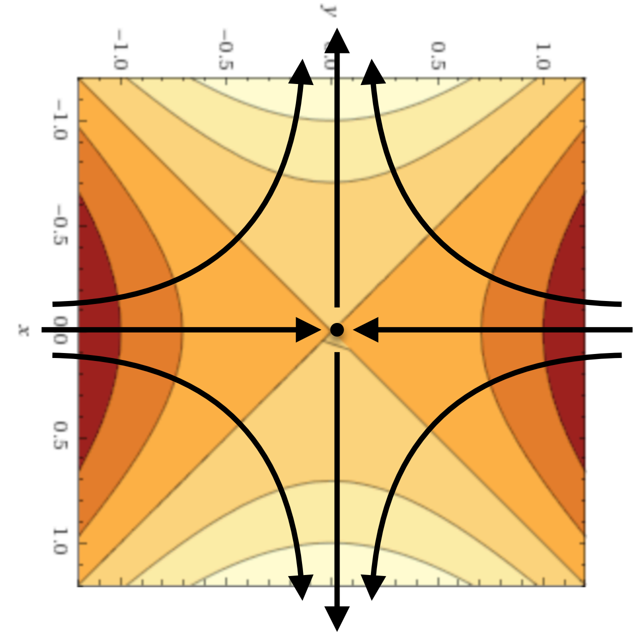
gradients are model systems: every dynamical system has 'gradient structure'

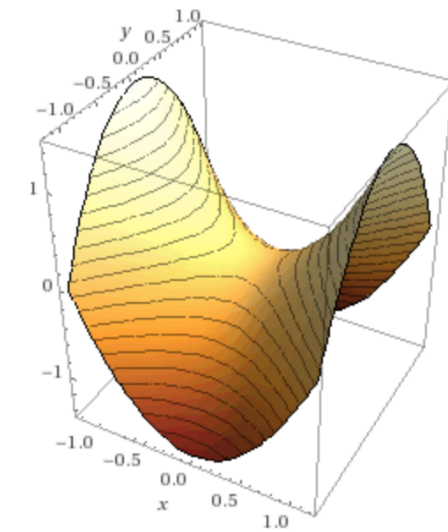
S. Smale, Differentiable dynamical systems, 1967 C. Conley, The gradient structure of a flow I, 1988

Morse indices measure fixed points

Morse index quantifies instability

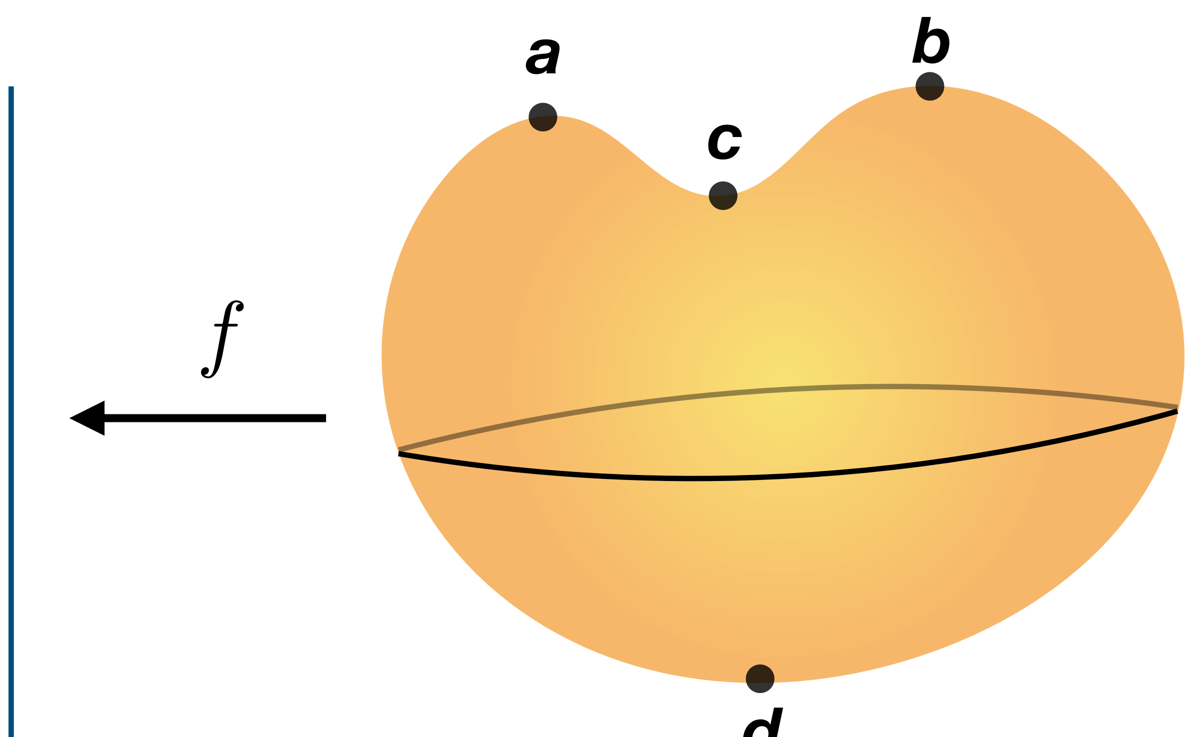
dimension of $W^u(p)$





Morse indices as minimal chain complex (zero differentials)

$$M_n(p) = \begin{cases} \mathbb{Z}_2\langle p\rangle, & n = \dim W^u(p) \\ 0, & \text{else} \end{cases}$$



$$\dot{x} = -\nabla f(x)$$

Morse indices assemble

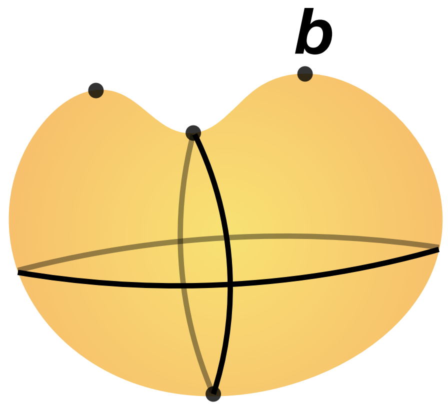
$$M_\bullet(f) = \bigoplus_{p \in crit(f)} M_\bullet(p)$$

$$0 \longrightarrow \mathbb{Z}_2\langle a\rangle \oplus \mathbb{Z}_2\langle b\rangle \xrightarrow{\begin{pmatrix}1 & 1\end{pmatrix}} \mathbb{Z}_2\langle c\rangle \xrightarrow{0} \mathbb{Z}_2\langle d\rangle \longrightarrow 0$$

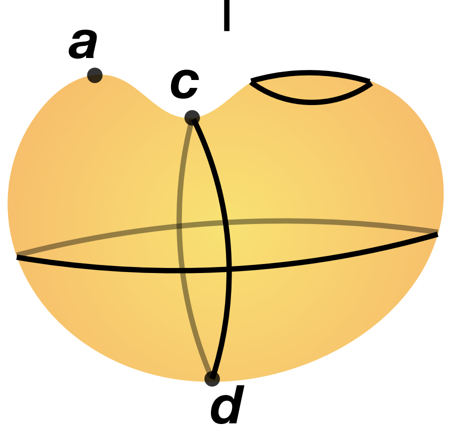
boundary operator counts connecting orbits mod 2

a height function filters

(via lattice of sublevel sets)



$$0 \leftarrow \mathbb{Z}_2\langle d\rangle \xleftarrow{0} \mathbb{Z}_2\langle c\rangle \xleftarrow{\begin{pmatrix}1 & 1\end{pmatrix}} \mathbb{Z}_2\langle a\rangle \oplus \mathbb{Z}_2\langle b\rangle \leftarrow 0$$



$$0 \leftarrow \mathbb{Z}_2\langle d\rangle \xleftarrow{0} \mathbb{Z}_2\langle c\rangle \xleftarrow{1} \mathbb{Z}_2\langle a\rangle \leftarrow 0$$

simple dynamics:
non-degenerate equilibria
heteroclinic orbits

$$f^{-1}(-\infty, x] \rightsquigarrow \{\mathbb{Z}_2\langle a\rangle : f(a) \leq f(x)\}$$

sublevel set (Morse) subcomplex

Morse index of ***b*** recovered
as a subquotient

$$M_\bullet(b) = \frac{0 \leftarrow \mathbb{Z}_2\langle d\rangle \xleftarrow{0} \mathbb{Z}_2\langle c\rangle \xleftarrow{\begin{pmatrix}1 & 1\end{pmatrix}} \mathbb{Z}_2\langle a\rangle \oplus \mathbb{Z}_2\langle b\rangle \leftarrow 0}{0 \leftarrow \mathbb{Z}_2\langle d\rangle \xleftarrow{0} \mathbb{Z}_2\langle c\rangle \xleftarrow{1} \mathbb{Z}_2\langle a\rangle \leftarrow 0}$$

$$0 \leftarrow 0 \leftarrow 0 \leftarrow \mathbb{Z}_2\langle b\rangle \leftarrow 0 = M_\bullet(b)$$

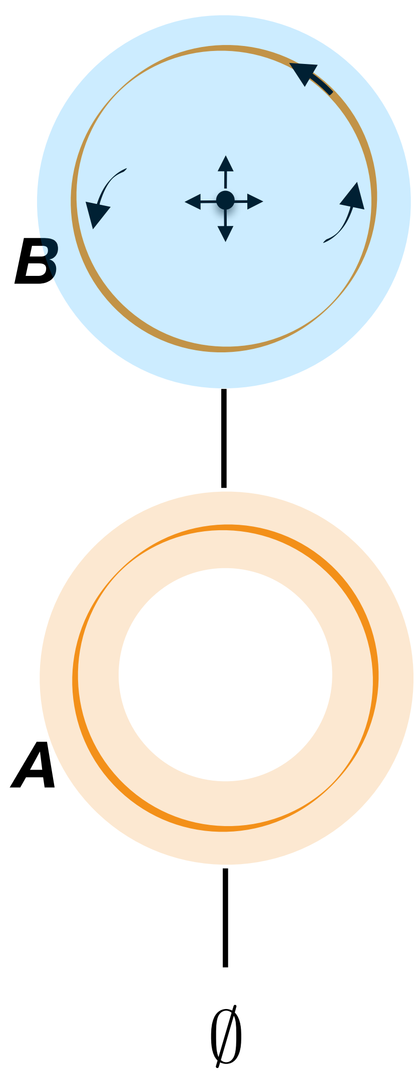
Conley's focus: attractors, attracting blocks

Conley theory is a purely topological generalization of Morse theory for general dynamical systems

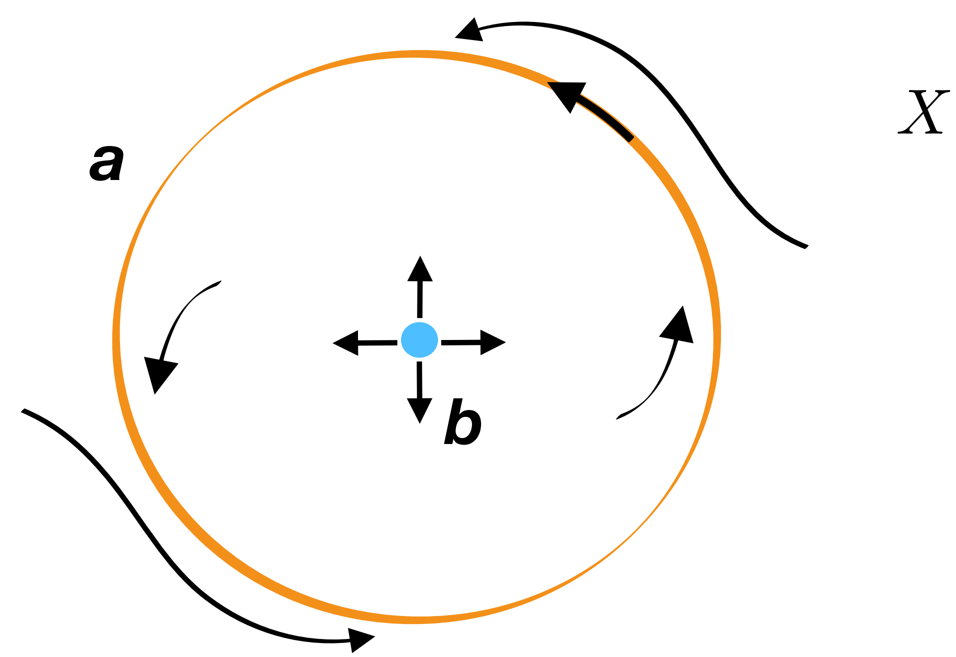
X compact metric space

dynamics given by continuous flow $\varphi : \mathbb{R} \times X \to X$

a compact set *N* is an attracting block if $\varphi(t, N) \subset \mathrm{int}(N)$ *for all t > 0*



L *sublattice of attracting blocks*



Fact: the set of attracting blocks **ABlock** is bounded distributive lattice

$\wedge := \cap \quad \vee := \cup$

Birkhoff's theorem

L finite distributive lattice

the poset of *join irreducible* elements of **L** is

$$\mathsf{J}(\mathsf{L}) := \{x \in \mathsf{L} \setminus \{0_L\} : \text{if } x = A \vee B, \text{ then } x = A \text{ or } x = B\}$$

a join-irreducible has a unique predecessor

$\mathsf{J}(\mathsf{L}) \ni B \mapsto \overleftarrow{B} \in \mathsf{L}$

$(\mathsf{P}, \leq)$ poset

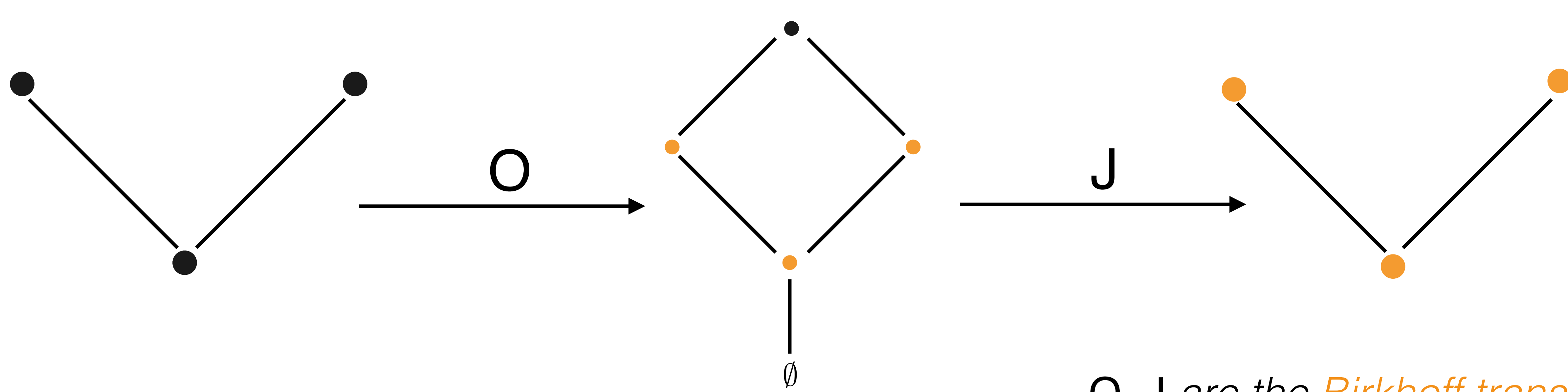
the lattice of lower sets is

$$\mathsf{O}(\mathsf{P}) := \{U \subseteq \mathsf{P} : \text{ if } x \in U \text{ and } y \leq x \text{ then } y \in U\}$$

$\wedge := \cap \quad \vee := \cup$

Fact: **O**, **J** are contravariant functors

Birkhoff: $\mathsf{O}(\mathsf{J}(\mathsf{L})) \cong \mathsf{L} \qquad \mathsf{J}(\mathsf{O}(\mathsf{P})) \cong \mathsf{P}$

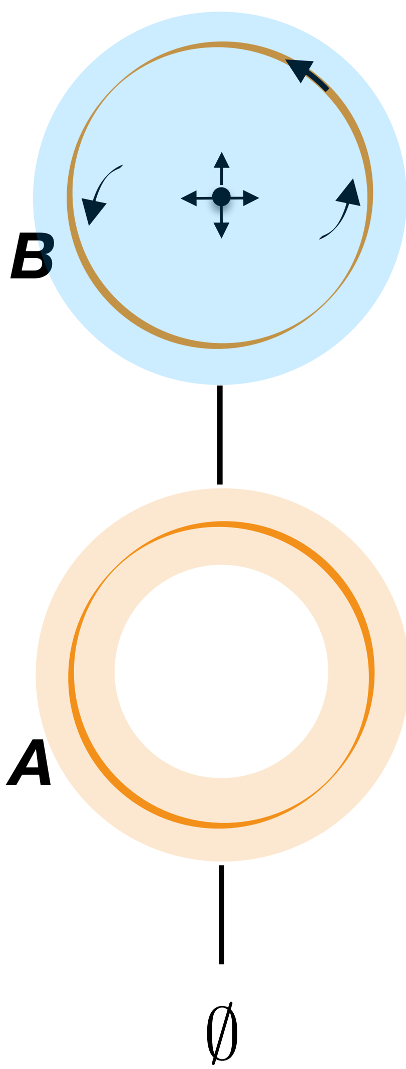


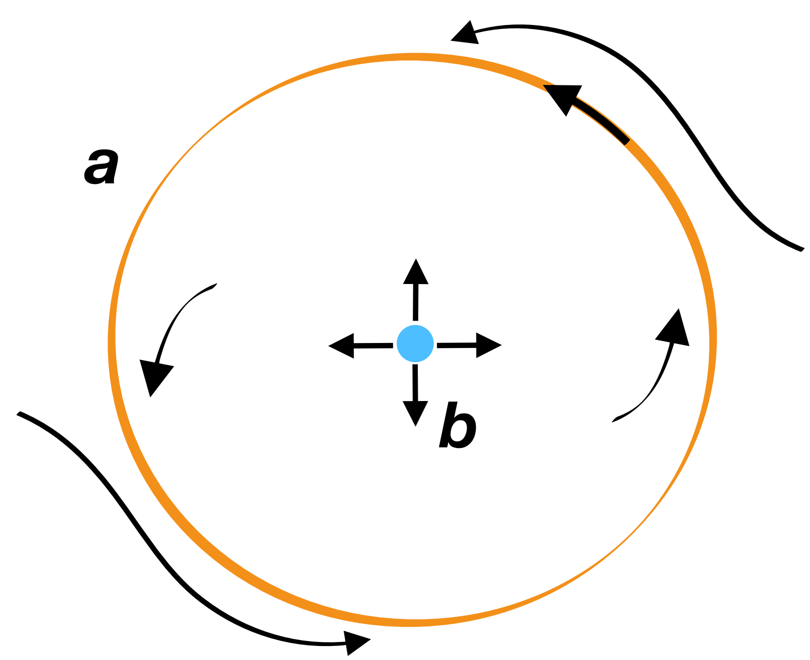
O, J *are the Birkhoff transforms*

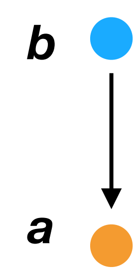
Conley-Morse Homology i

the Birkhoff transforms give dual perspective to dynamics

$$\begin{array}{ccc} \boldsymbol{B} & \longmapsfrom & \{a,b\} \\ \boldsymbol{A} & \longmapsfrom & \{a\} \end{array} \qquad \begin{array}{ccc} \mathsf{L} & \xleftarrow{\cong} & \mathsf{O(P)} \\ \bigcup_{p\in U} W^u(p) & \longmapsfrom & \boldsymbol{U} \end{array}$$







poset P
of isolated invariant sets
(Morse sets)

$$\begin{array}{ccc} \mathsf{J(L)} & \xrightarrow{\cong} & \mathsf{P} \\ \boldsymbol{B} & \longmapsto & \mathsf{Inv}(B \setminus \overleftarrow{B}) \end{array} \qquad \begin{array}{ccc} \boldsymbol{B} & \longmapsto & \boldsymbol{b} \\ \boldsymbol{A} & \longmapsto & \boldsymbol{a} \end{array}$$

L *sublattice of attracting blocks*

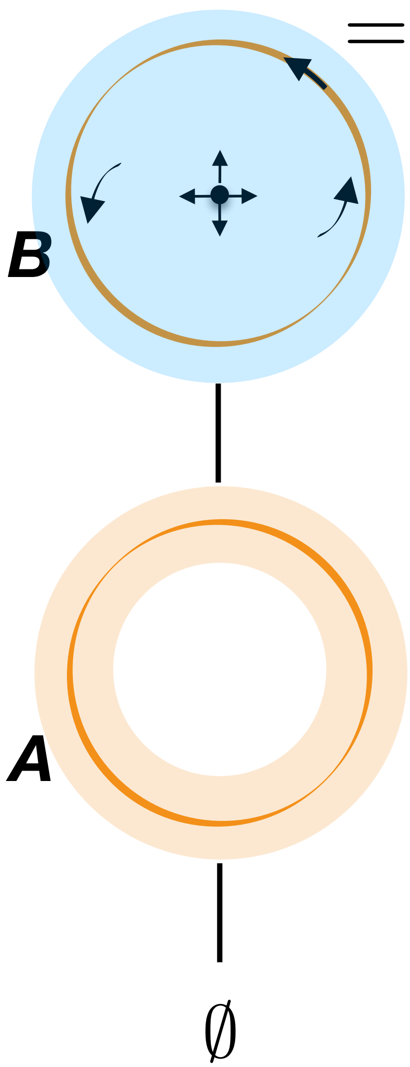
Conley-Morse Homology ii

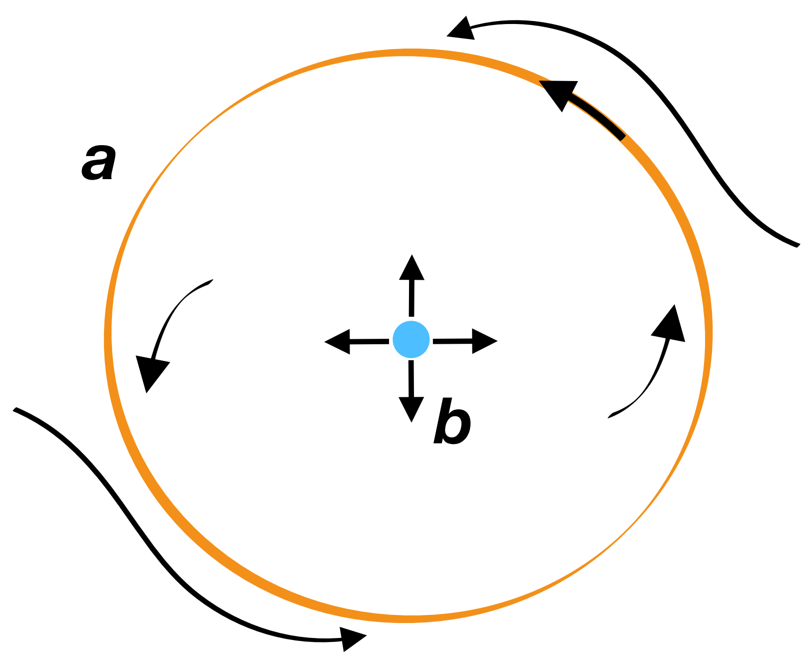
to generalize Morse homology

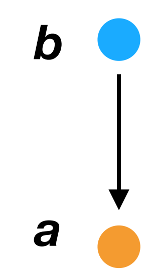
associate minimal complex to isolated invariant sets (Conley index)

characterized by dynamics at the boundary (local instability)

$$CH_\bullet(b) := H_\bullet(B, \overleftarrow{B}) = H_\bullet(B, A)$$



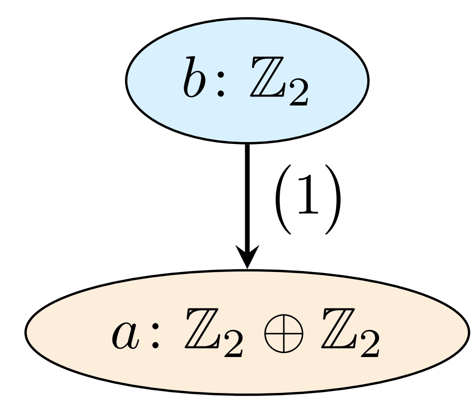




poset P
of isolated invariant sets

chain complex of Conley indices

$$0 \leftarrow \mathbb{Z}_2\langle a\rangle \xleftarrow{0} \mathbb{Z}_2\langle a\rangle \xleftarrow{1} \mathbb{Z}_2\langle b\rangle \leftarrow 0$$



boundary operator is called the connection matrix

L *sublattice of attracting blocks*

Franzosa, Mischaikow, McCord, Reineck… Conley-Morse homology is a homology theory

Conley-Morse Homology iii

to generalize Morse homology

Conley indices as input to chain complex

what is the boundary operator?

for **L** sublattice of attracting blocks and **J(L)** poset of Morse sets

Theorem (Franzosa, Robbin & Salamon): There exists a strictly upper triangular - wrt $(\mathsf{J}(\mathsf{L}), \leq)$ - boundary operator

$$\Delta : \bigoplus_{p \in \mathsf{J}(\mathsf{L})} CH_\bullet(p) \to \bigoplus_{p \in \mathsf{J}(\mathsf{L})} CH_\bullet(p)$$

so that for any attracting block A in **L** the homology of

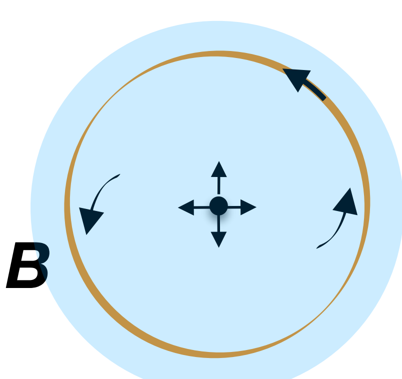
local to global

$$\Delta : \bigoplus_{\{p \in \mathsf{J}(\mathsf{L}) :\ p \leq A\}} CH_\bullet(p) \to \bigoplus_{\{p \in \mathsf{J}(\mathsf{L}) :\ p \leq A\}} CH_\bullet(p)$$

is isomorphic to $H_\bullet(A)$

algebraic representation of dynamics

Δ is called a *connection matrix*



B

$$H_\bullet(B) \cong H_\bullet\big(CH_\bullet(a) \oplus CH_\bullet(b), \Delta\big)$$

$$b : \mathbb{Z}_2 \xrightarrow{(1)} a : \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

$$\Delta = \begin{array}{cc|ccc} & & a & a & b \\ & & 0 & 1 & 2 \\ \hline a & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 1 \\ b & 2 & 0 & 0 & 0 \end{array}$$

valuation
cell dim.

computational Conley theory

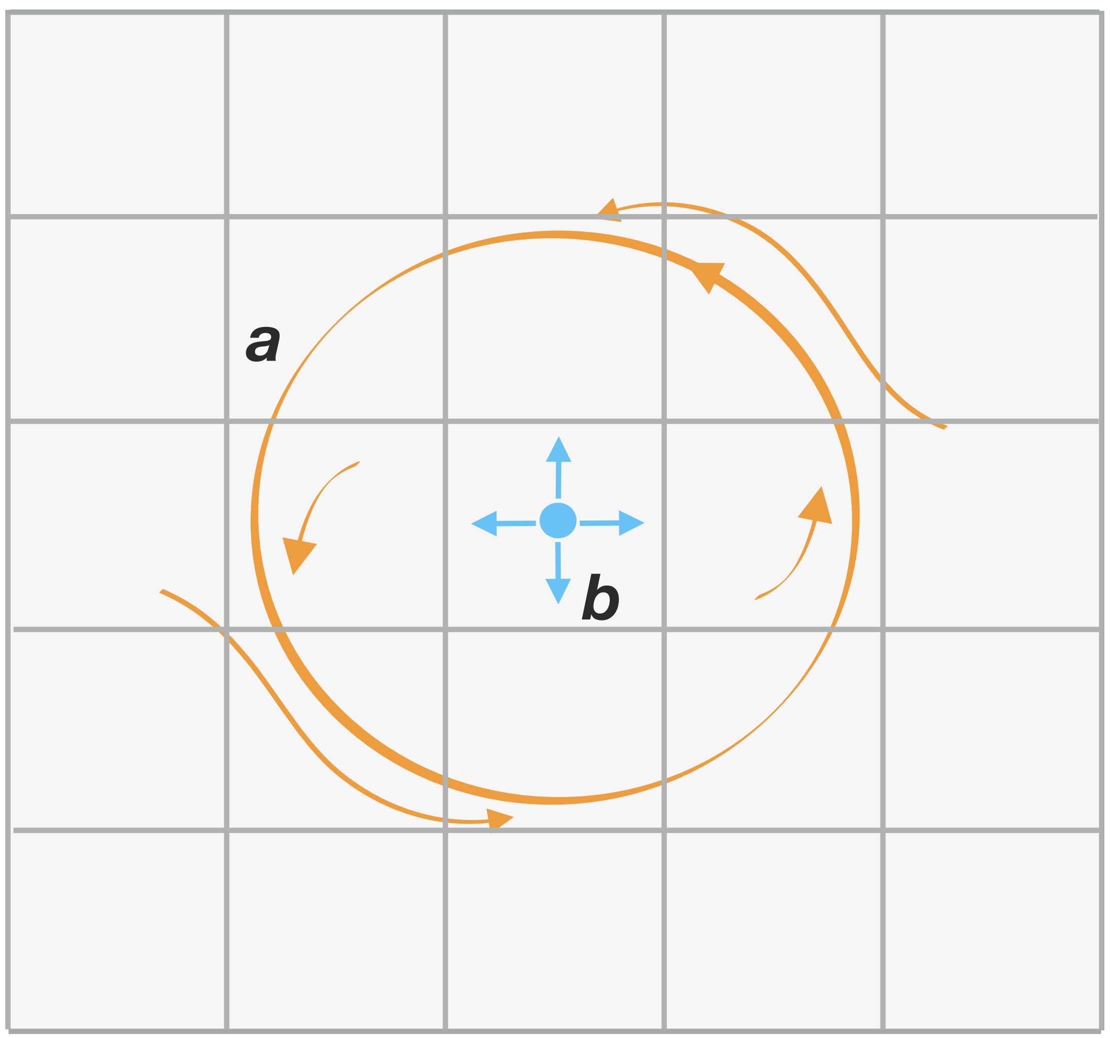
approximation + data structures

topological spaces are approximated

with cell complexes (cubical, simplicial, polyhedral)

$X \subset \mathbb{R}^2$

approximation by cubical complex



Definition (Cell complex)

$\mathcal{X} = (\mathcal{X}, \leq, \kappa, \dim)$ consists of poset $(\mathcal{X}, \leq)$

with two functions $\dim : \mathcal{X} \to \mathbb{N}$ and $\kappa : \mathcal{X} \times \mathcal{X} \to k$

satisfying:

1. **dim** is a poset morphism;
2. for all $\xi, \xi' \in \mathcal{X}$

$$\kappa(\xi, \xi') \neq 0 \implies \xi' \leq \xi \text{ and } \dim(\xi) = \dim(\xi') + 1$$

3. for all $\xi, \xi' \in \mathcal{X}$

$$\sum_{\xi'' \in \mathcal{X}} \kappa(\xi, \xi'') \cdot \kappa(\xi'', \xi') = 0$$

a cell complex $\mathcal{X}$ generates

a chain complex $(C_\bullet(\mathcal{X}), \partial)$

if the attracting blocks are representable by subcomplexes

then we may compute algebraic invariants, e.g. homology

X cell complex Sub(X) lattice of subcomplexes of X

representation of attracting blocks

inclusion is a lattice homomorphism

$$L \xrightarrow{\;i\;} \mathrm{Sub}(X)$$

L *sublattice of attracting blocks*

Sub(X) *lattice of subcomplexes*

the Conley indices can be computed for both attractors…

Betti numbers

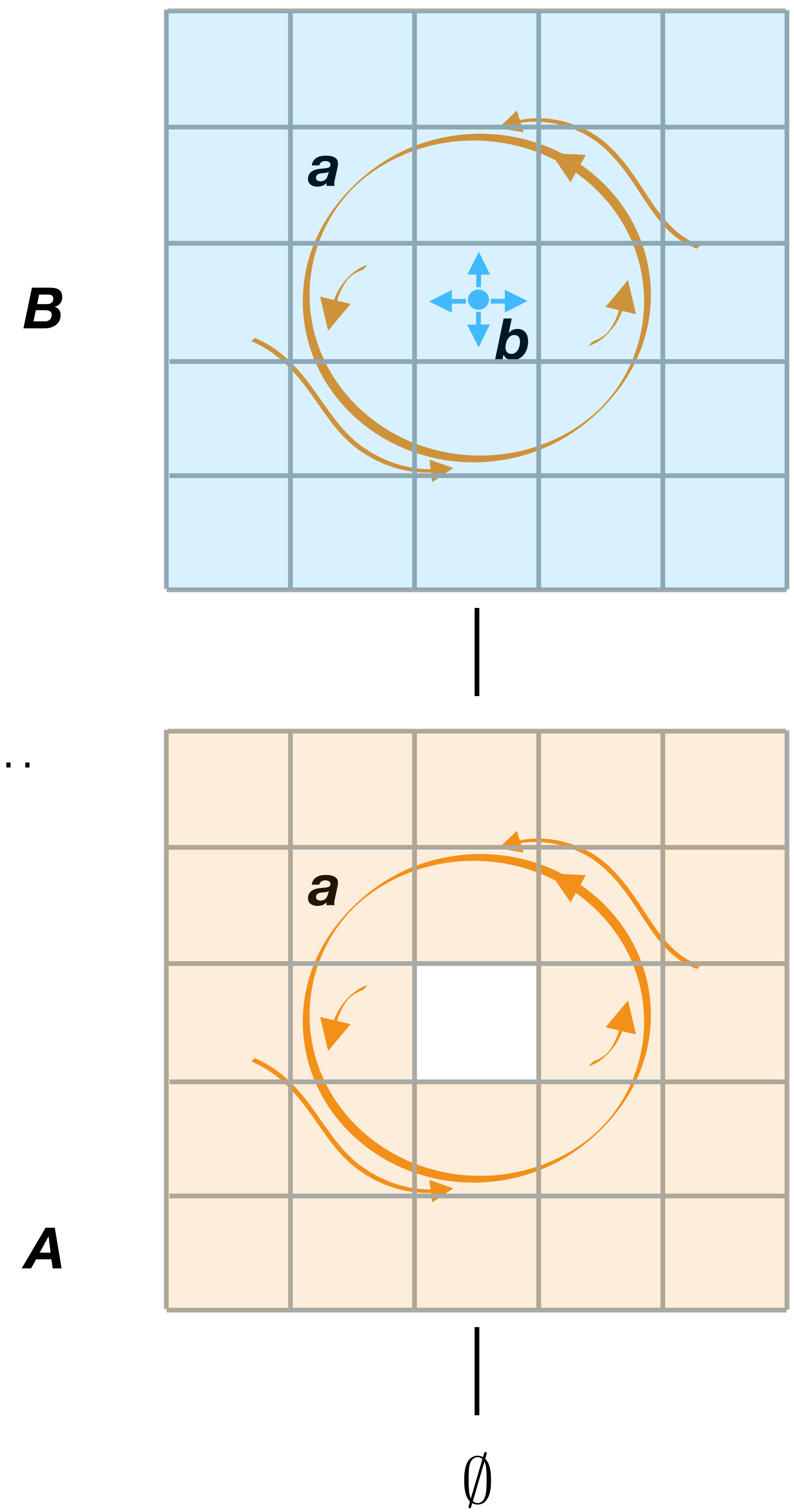
$CH_\bullet(A) = H_\bullet(A)$ = (1, 1, 0)

$CH_\bullet(B) = H_\bullet(B)$ = (1, 0, 0)

… and the invariant sets (Morse sets)

$CH_\bullet(a) = H_\bullet(A, \emptyset)$ = (1, 1, 0)

$CH_\bullet(b) = H_\bullet(B, A)$ = (0, 0, 1)



What is the data structure for Conley theory?

L sublattice of attracting blocks

Sub(X) lattice of subcomplexes of X

inclusion is lattice morphism

$$\mathsf{L} \xrightarrow{\;i\;} \mathsf{Sub(X)}$$

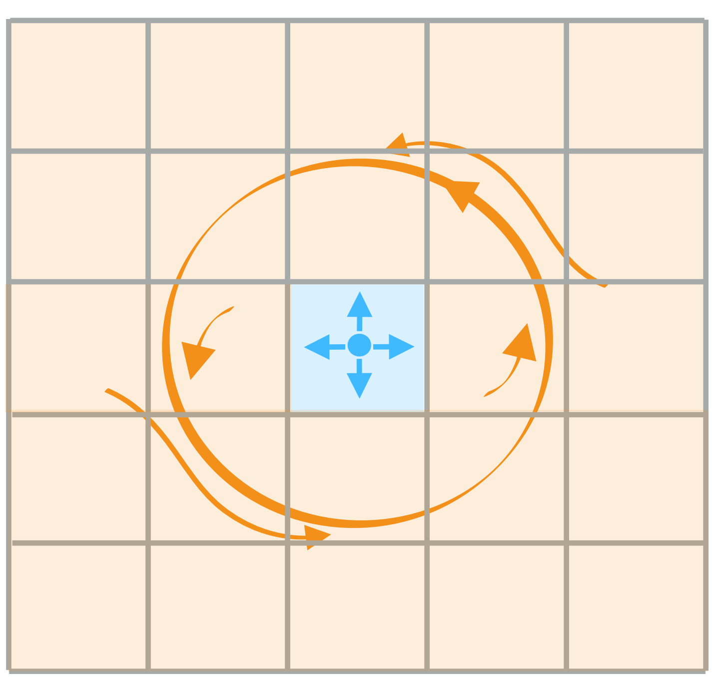
…applying the Birkhoff transform (contravariant)

(X, $\leq$) face poset

J(Sub(X)) = (X, $\leq$)

J(L) poset of join-irreducibles

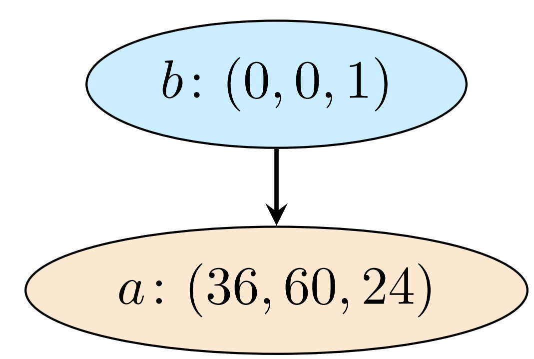
$$(\mathsf{X}, \leq) \xrightarrow{\;\mathsf{J}(i)\;} (\mathsf{J(L)}, \leq)$$



$$\xrightarrow{\;\mathsf{J}(i)\;}$$

poset morphism

count of cells in the fiber $\mathsf{J}(i)^{-1}(b)$ for each dim.



Axiomization (Conley theoretic data structure)

X cell complex

(X, $\leq$) face poset

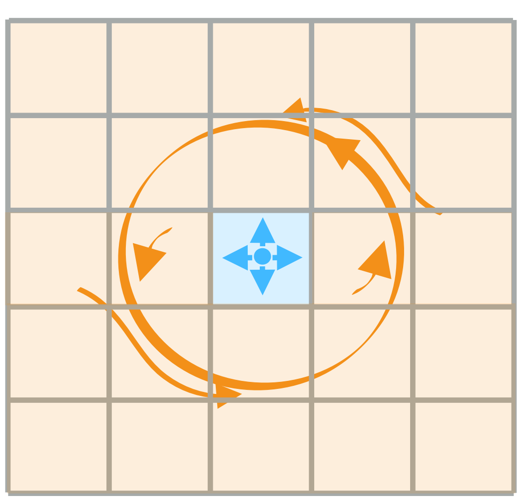
P poset

Definition (P-graded cell complex)

X, P, and a poset morphism ν from X to P

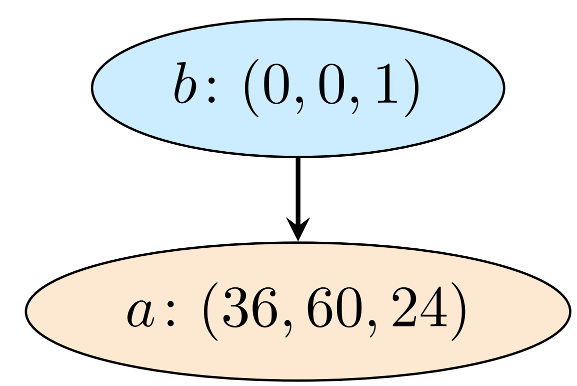
$$(\mathsf{X}, \leq) \xrightarrow{\nu} (\mathsf{P}, \leq)$$

topological approximation



$\xrightarrow{\nu}$

dynamical information



a graded cell complex determines

a **P**-graded chain complex $(C(\mathsf{X}), \partial)$

$$C(\mathsf{X}) = \bigoplus_{p \in \mathsf{P}} C(\nu^{-1}(p))$$

boundary map is **P**-graded

$$\partial_{pq} \neq 0 \implies p \leq q$$

upper triangular wrt **P**

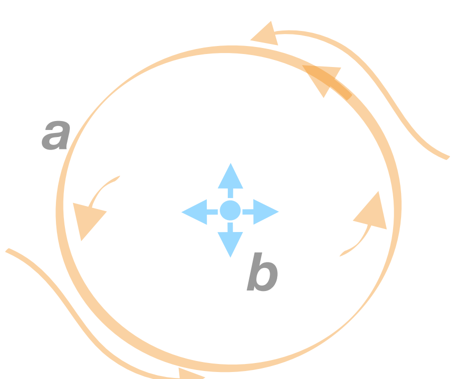
Axiomization (Connection matrix)

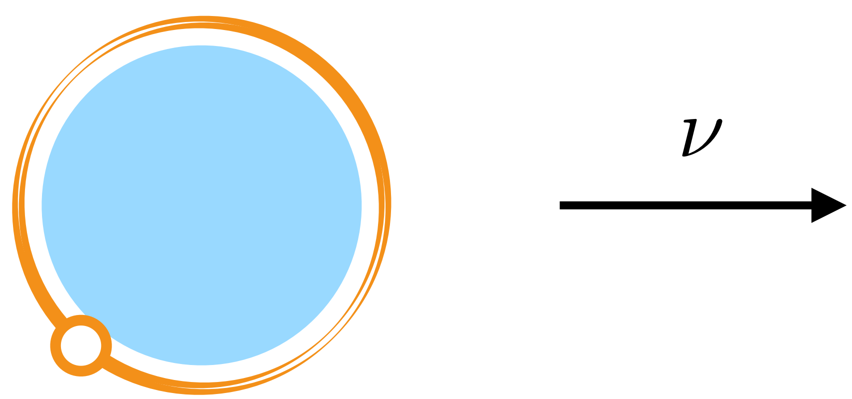
Definition (strict **P**-graded complex)

P-graded complex with minimal fibers

$\partial_{pp} = 0$ for ***p*** in **P** ‘small’ objects

i.e. ∂ is strictly upper triangular wrt **P**





Betti numbers of fiber
(Conley index)

$$\begin{tikzcd} b : (0,0,1) \arrow[d] \\ a : (1,1,0) \end{tikzcd}$$

goal: replace graded complex with equivalent strict graded complex

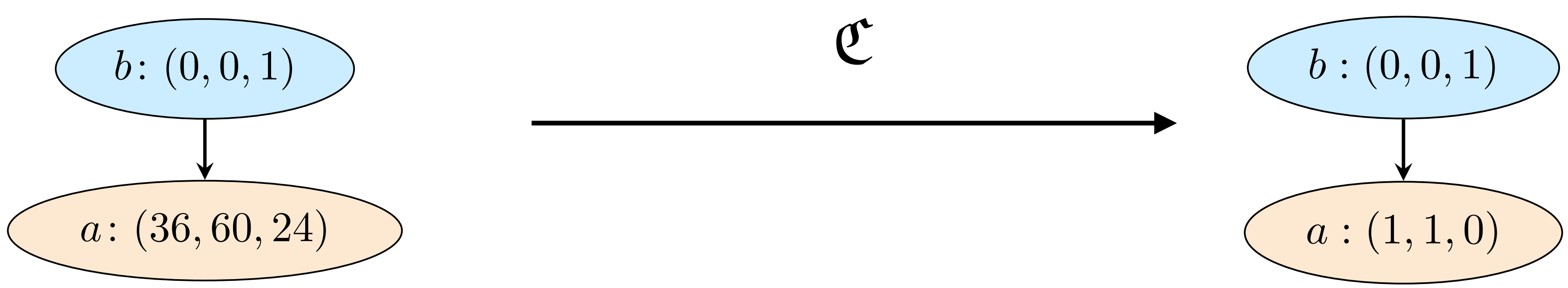
interpretation of connection matrix for computation:

a *Conley complex* is a strict representative of graded chain equivalence class

the boundary operator of a Conley complex is a *connection matrix*

Theorem: there is a functor $\mathfrak{C}$ taking a graded complex to a Conley complex

Betti numbers of fiber
(Conley index)



'small' object

under the hood: discrete morse theory
homotopy categories

computational perspective: chain-level data reduction
without loss of homological information

Harker + Mischaikow + S.

computational Conley homology

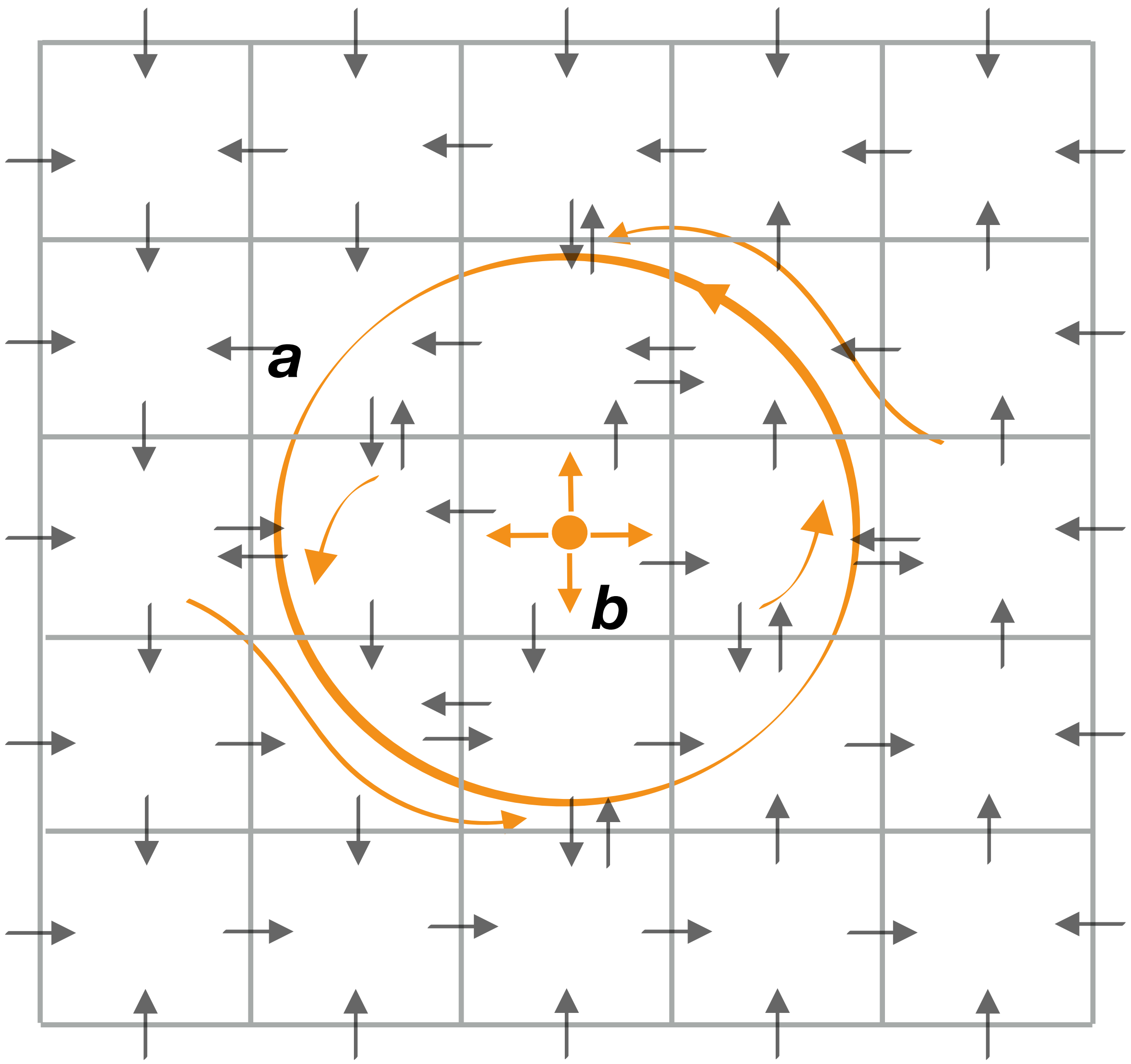
applications + implementation

application i:

state transition models

topological spaces are approximated
with cell complexes

continuous dynamics are approximated
with directed graph $\mathcal{F}$ on top cells $\mathcal{X}^+$



lattice of forward invariant sets:

$$\mathsf{Invset}^+(\mathcal{F}) := \{U \subset \mathcal{X}^+ : \mathcal{F}(U) \subset U\}$$

poset of strongly connected components:

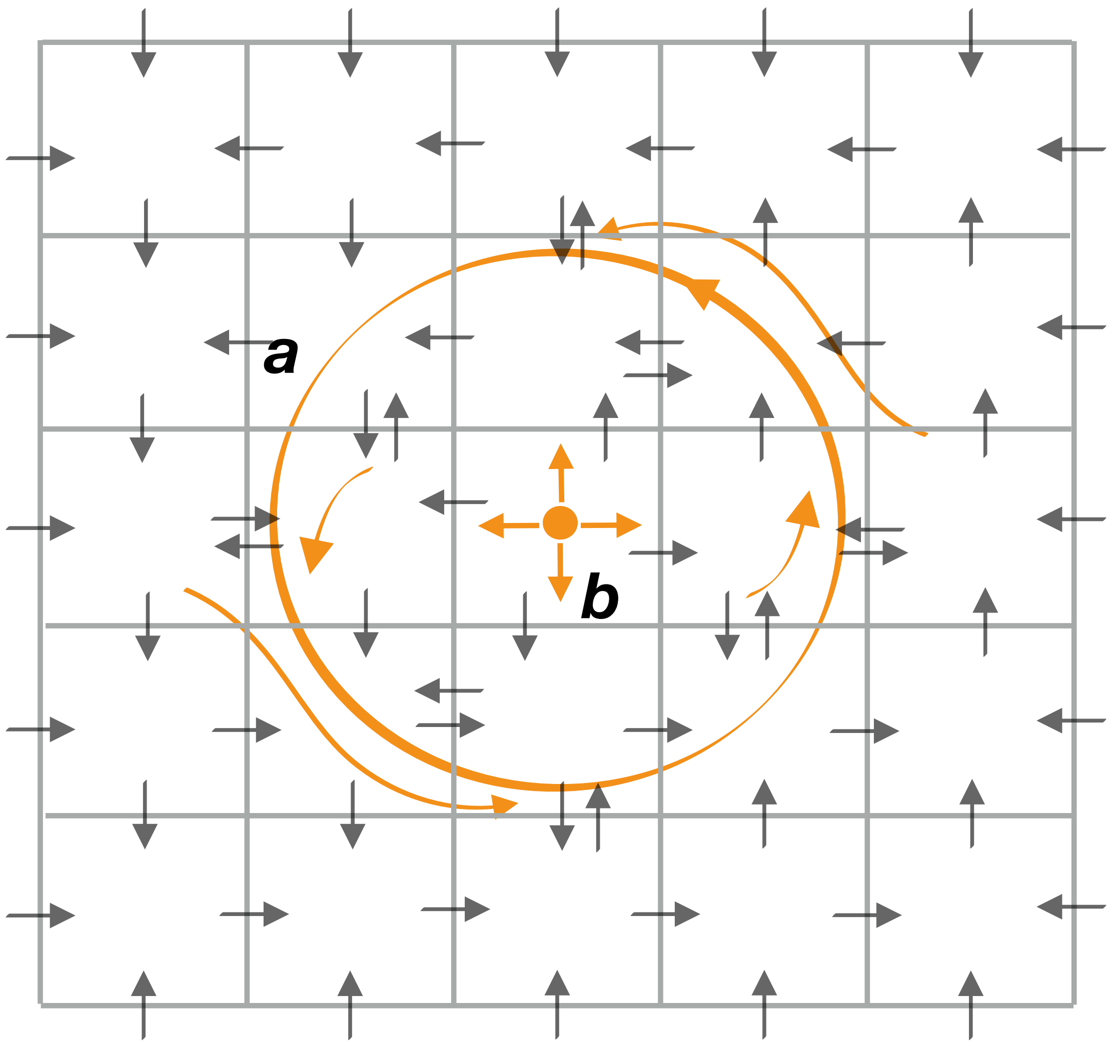
$$\mathsf{SC}(\mathcal{F}) := \mathsf{J}(\mathsf{Invset}^+(\mathcal{F}))$$

maximal recurrent sets of graph

poset $\mathsf{SC}(\mathcal{F})$ strongly connected components of $\mathcal{F}$

$$\mathcal{X}^+ \xrightarrow[\xi \mapsto [\xi]]{} \mathsf{SC}(\mathcal{F})$$

state transition models

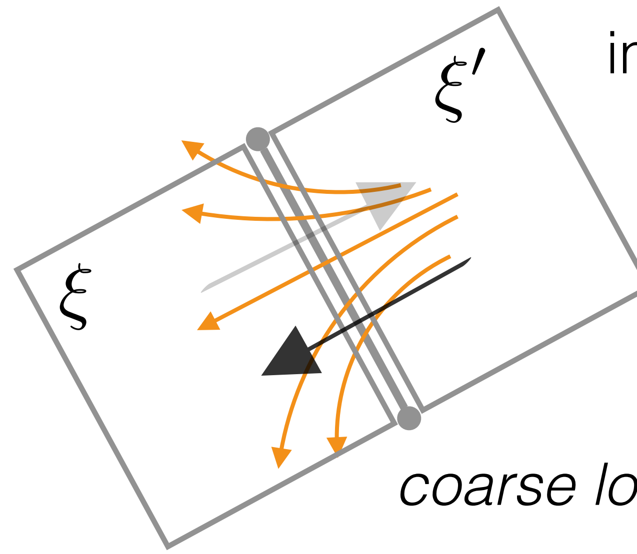


transversality condition

there is an edge $\xi \to \xi'$ between adjacent top cells

unless flow is transverse to $\mathrm{cl}(\xi) \cap \mathrm{cl}(\xi')$

in the direction $\xi' \to \xi$



coarse lower bound on dynamics

Theorem: if the graph is a state transition model then there is an extension ν

$$\begin{tikzcd} \mathcal{X}^+ \arrow[r] \arrow[d] & (\mathcal{X}, \leq) \arrow[dl, dashed, "\nu"] \\ \mathsf{SC}(\mathcal{F}) & \end{tikzcd}$$

graded cell complex

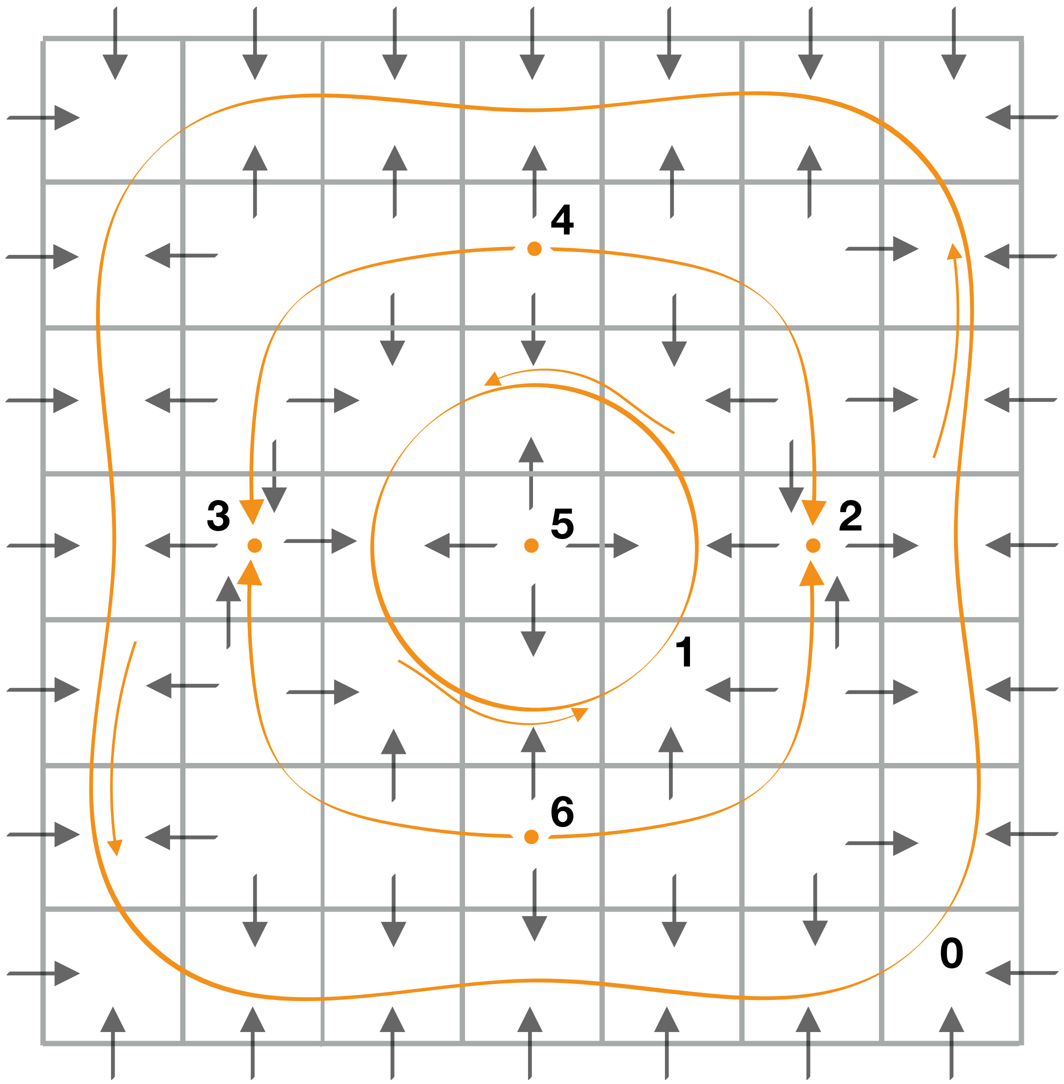
1. $\mathsf{A} = \{\nu^{-1}(a)\}_{a \in \mathsf{O}(\mathsf{SC}(\mathcal{F}))}$ is a lattice of attracting blocks for φ

2. $\mathfrak{C}(\mathcal{X}, \nu)$ is a Conley complex for φ

Remark: Computations + theorems are valid for any differential equation which is transverse to top cell boundaries in direction indicated

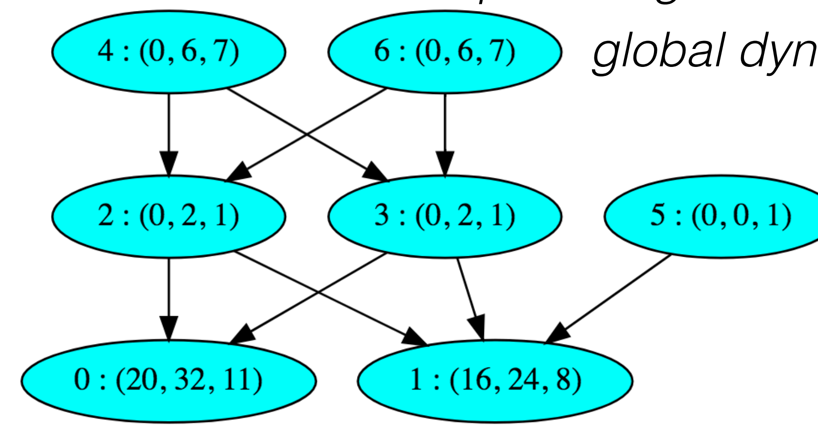
Harker + Mischaikow + S. + Vandervorst

state transition models ii



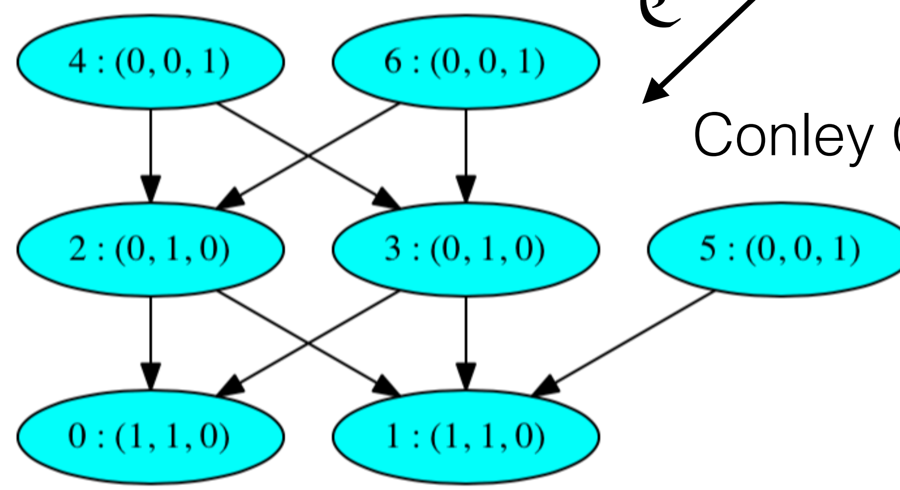
SC($\mathcal{F}$)-graded
cell complex

poset organizes the
global dynamics



$\mathfrak{C}$

Conley Complex



connection matrix is represented
with respect to a basis

different bases give different
qualitative descriptions of dynamics

in this example: four different bases

$$
\Delta_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \begin{pmatrix} & & 1 \\ & 1 & 1 \\ 1 & & 1 \\ 1 & & 1 \end{pmatrix} \end{array}
$$

$\longleftrightarrow$

Connection Matrix Data
==================
Boundaries of 0-cells (by cell index):
Cell 0 (valuation 1) : {}
Cell 1 (valuation 0) : {}
Boundaries of 1-cells (by cell index):
Cell 2 (valuation 2) : {0, 1}
Cell 3 (valuation 3) : {0, 1}
Cell 4 (valuation 0) : {}
Cell 5 (valuation 1) : {}
Boundaries of 2-cells (by cell index):
Cell 6 (valuation 6) : {2, 3, 4, 5}
Cell 7 (valuation 5) : {5}
Cell 8 (valuation 4) : {2, 3}

connection matrix as 'matrix'

connection matrix as data structure

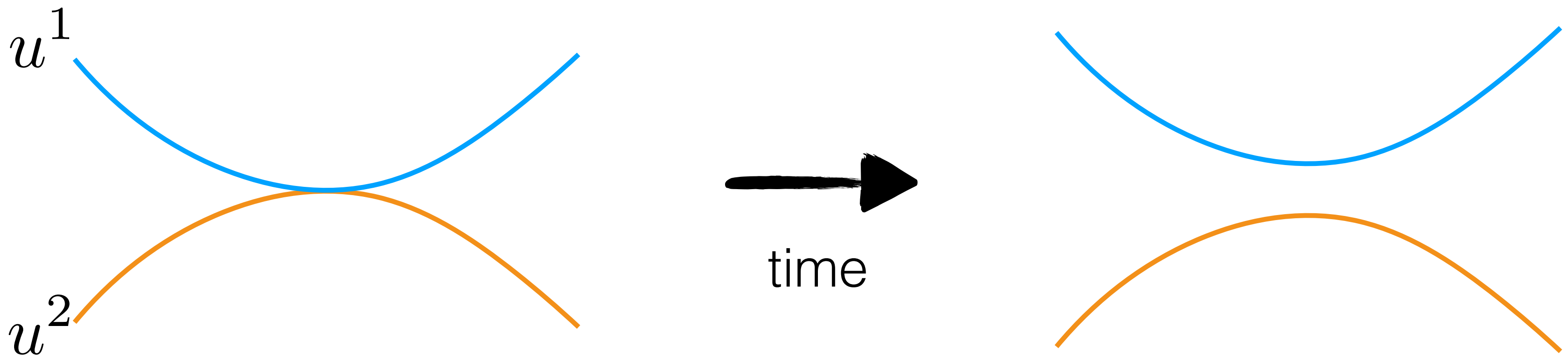
application ii:

Morse theory on spaces of braids

instantiation: dynamics on braids

$$u_t = u_{xx} + f(x, u, u_x)$$

parabolic dynamics decreases intersections



'comparison principle'

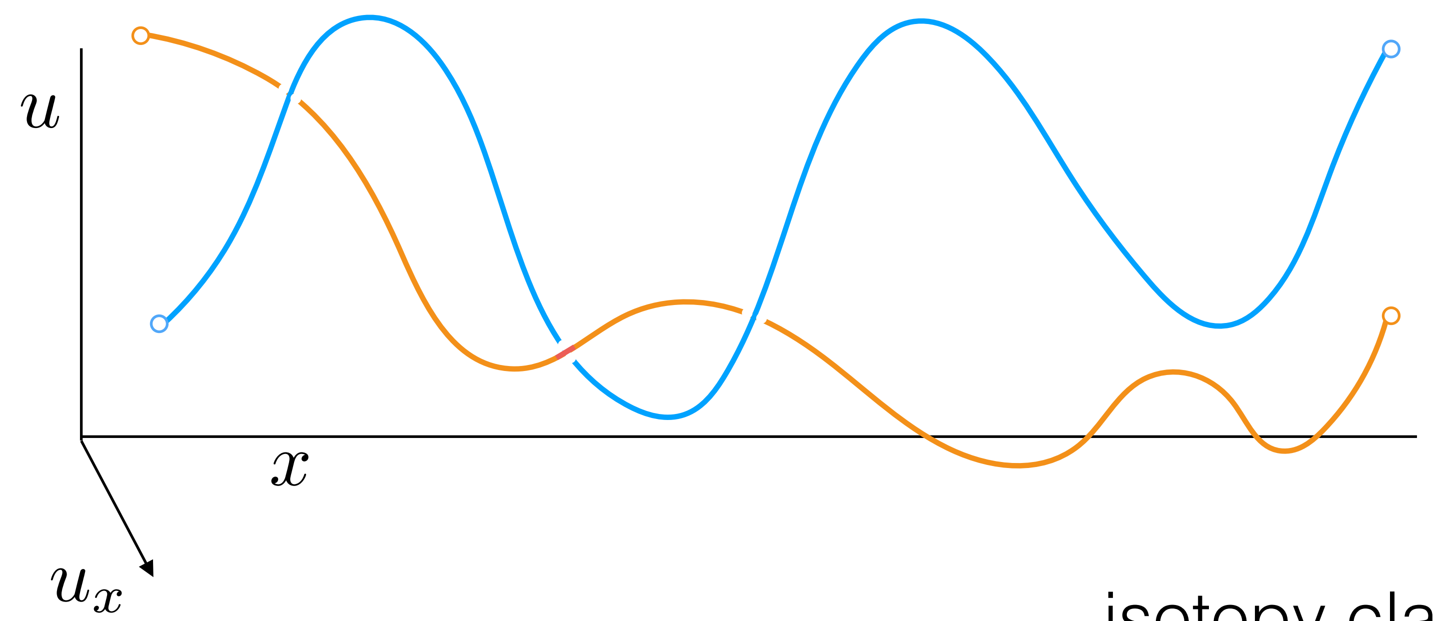
proof:

$$\frac{\partial}{\partial t}(u^1(x,t) - u^2(x,t)) = u^1_{xx} + f(x, u^1, 0) - u^2_{xx} - f(x, u^2, 0)$$

$$= u^1_{xx} - u^2_{xx} > 0$$

van den Berg, Ghrist, van der Vorst, Inventiones Math. 2003

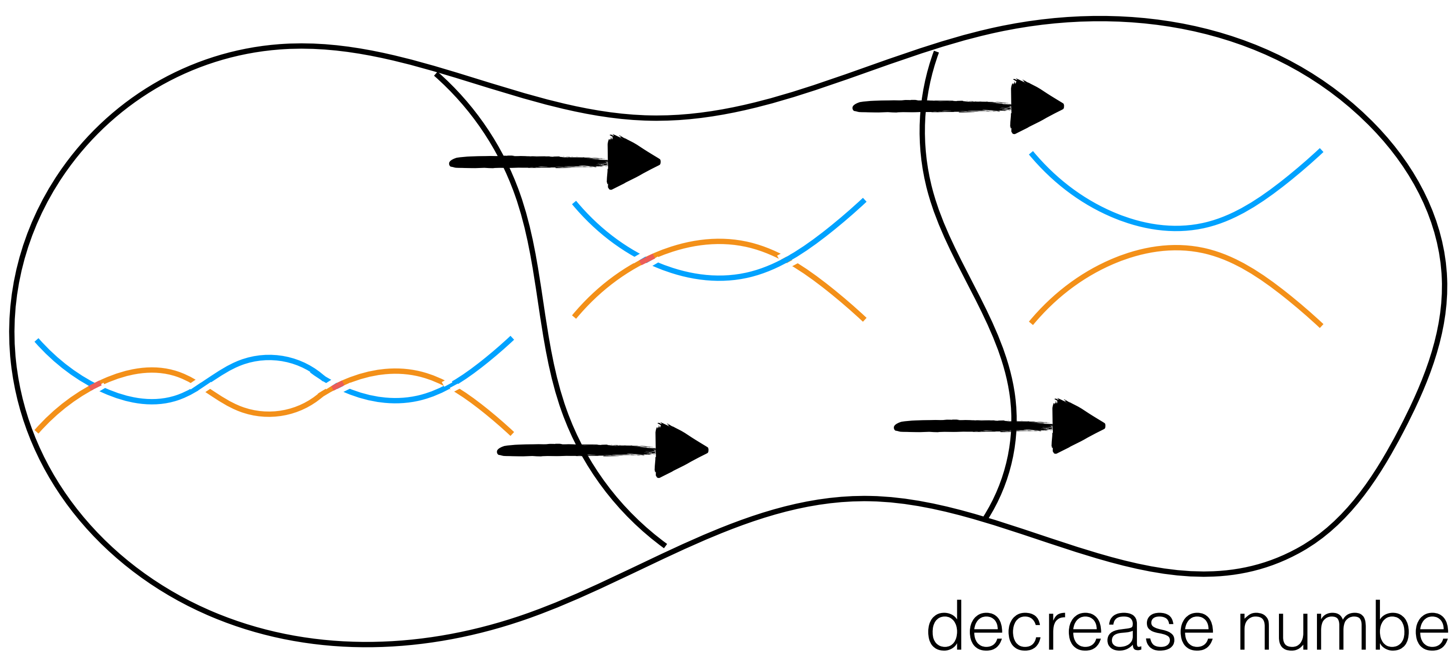
functions lift to braids



strands

isotopy class, fixed endpoints

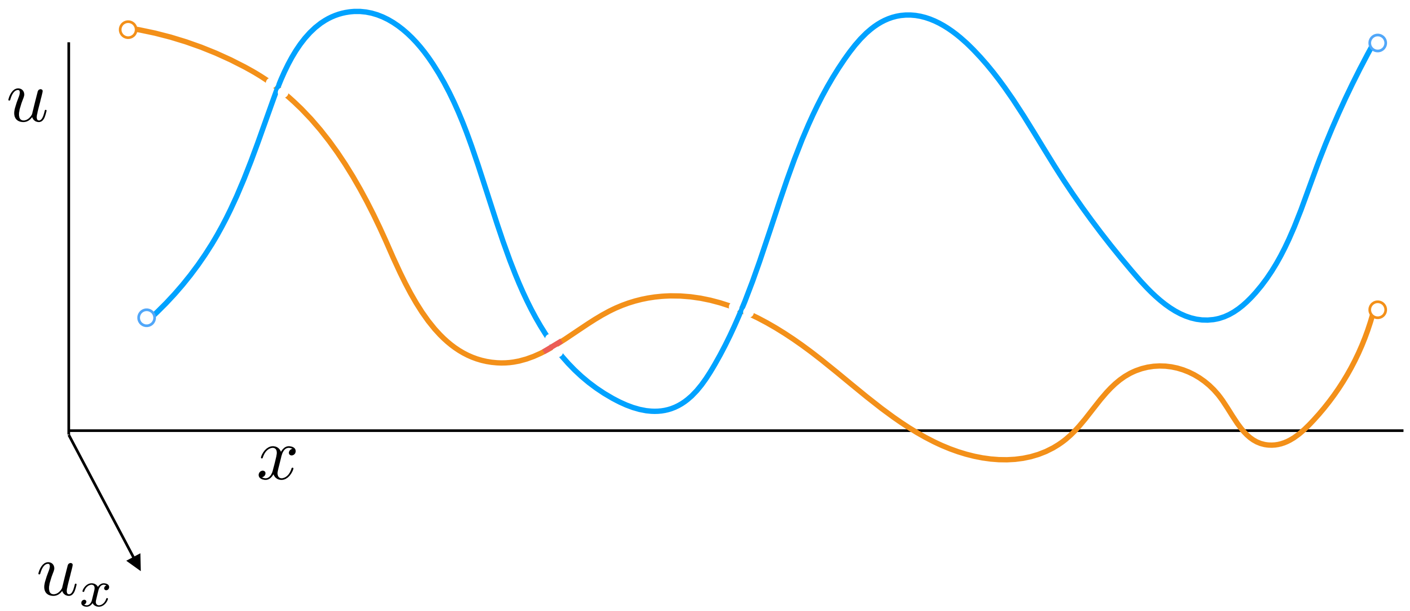
dynamics on braid classes



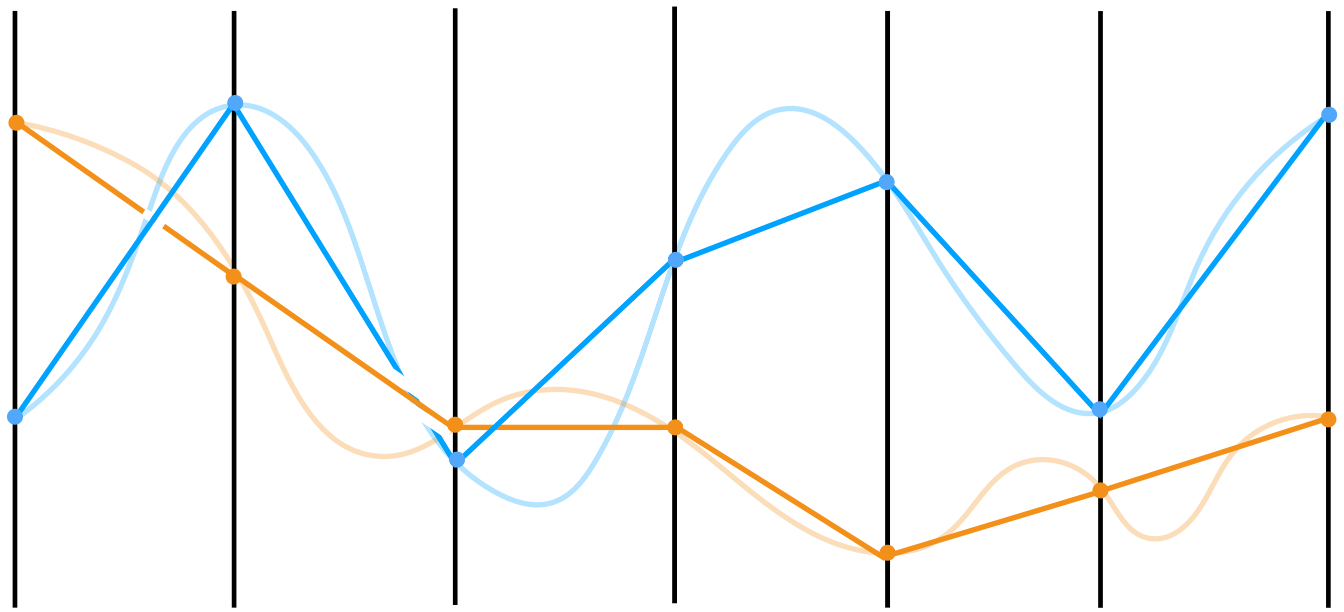
decrease number of intersections

van den Berg, Ghrist, van der Vorst,
Inventiones Math. 2003

functions lift to braids

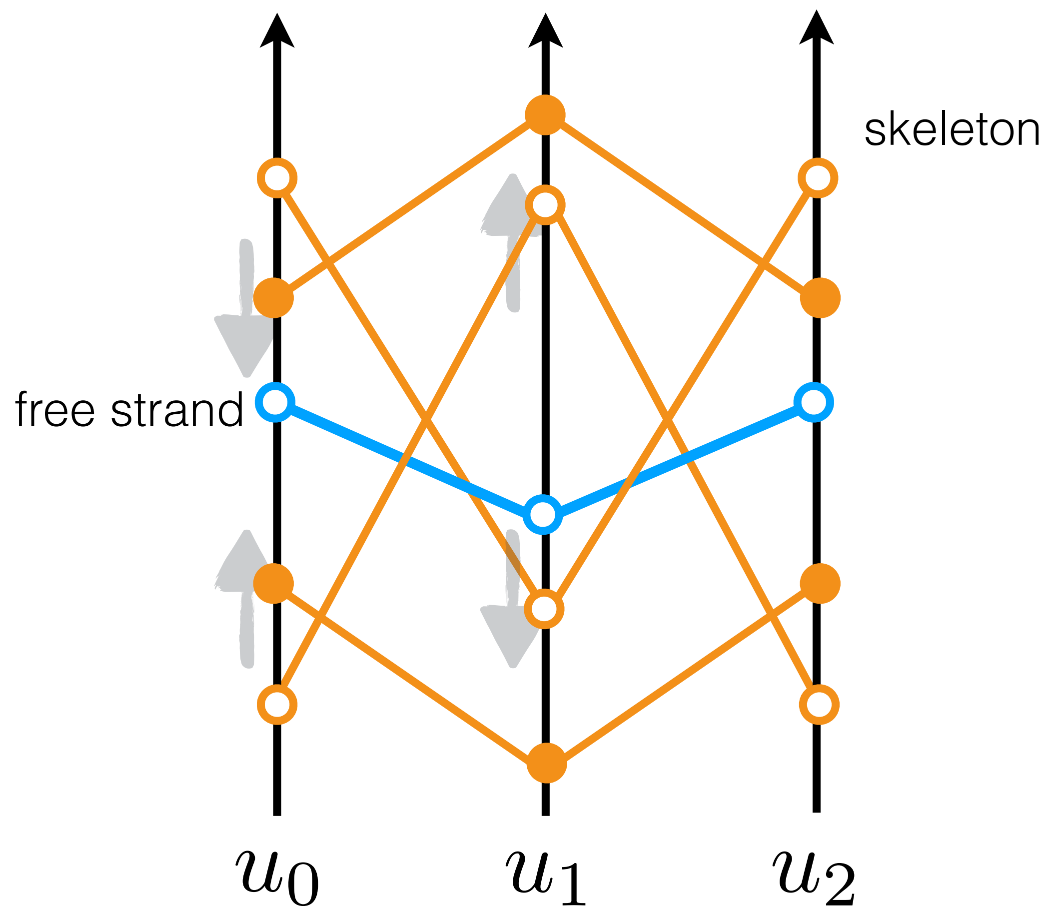


combinatorialization



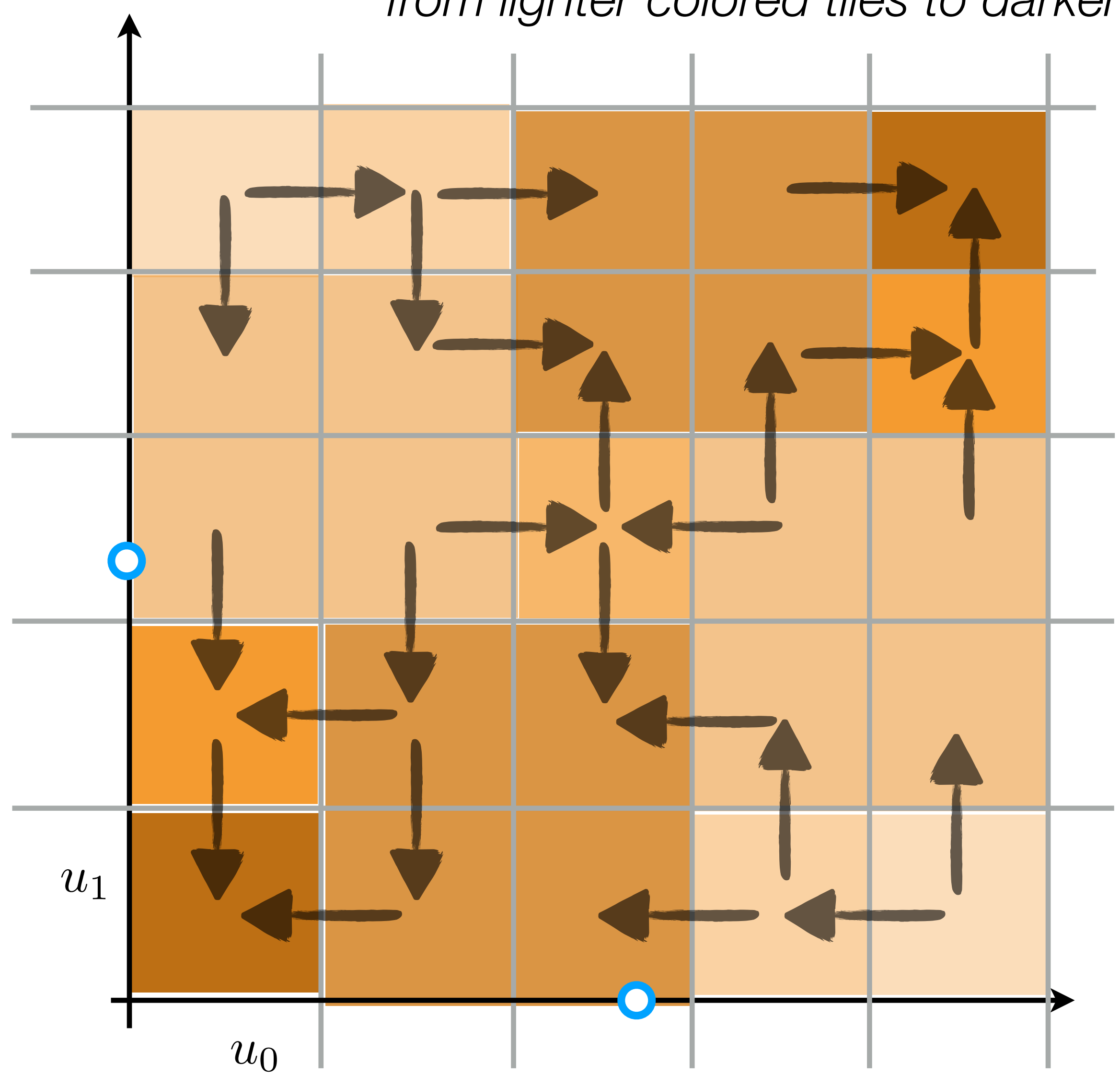
Morse theory on braids

van den Berg, Ghrist, van der Vorst,
Inventiones Math. 2003



Braided equilibrium solutions to parabolic PDE
with periodic boundary conditions

Solutions flow across boundary edges
from lighter colored tiles to darker



state transition model in $\mathbb{R}^2$

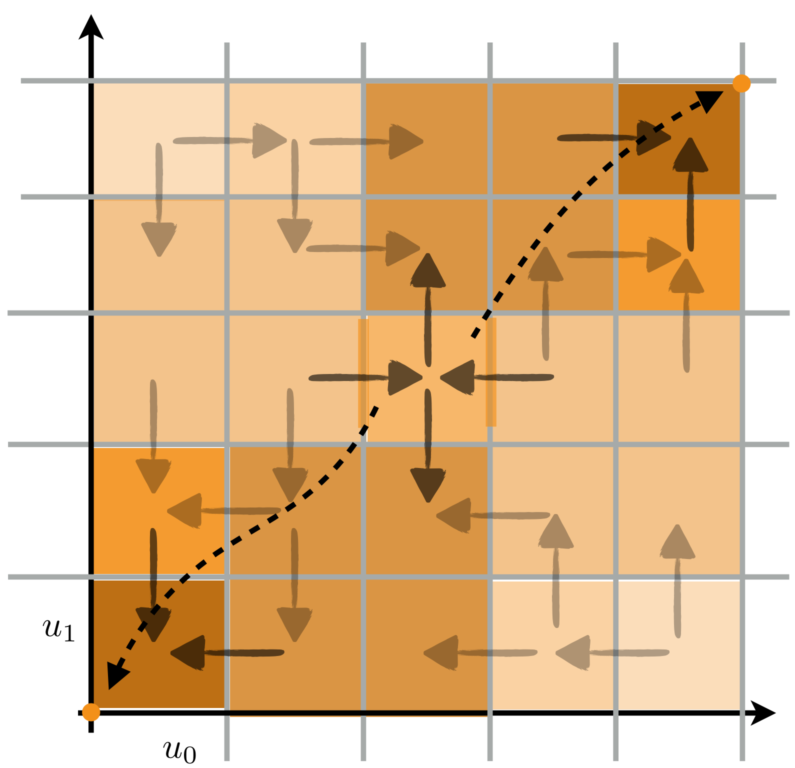
graded cubical complex in $\mathbb{R}^2$

Fact: Nontrivial Conley indices imply existence of solutions to PDE

Fact: Nonzero entry in connection matrix between adjacent elements proves existence of connecting orbit

braids i

$(X, \leq) \xrightarrow{\nu} SC(\mathcal{F})$

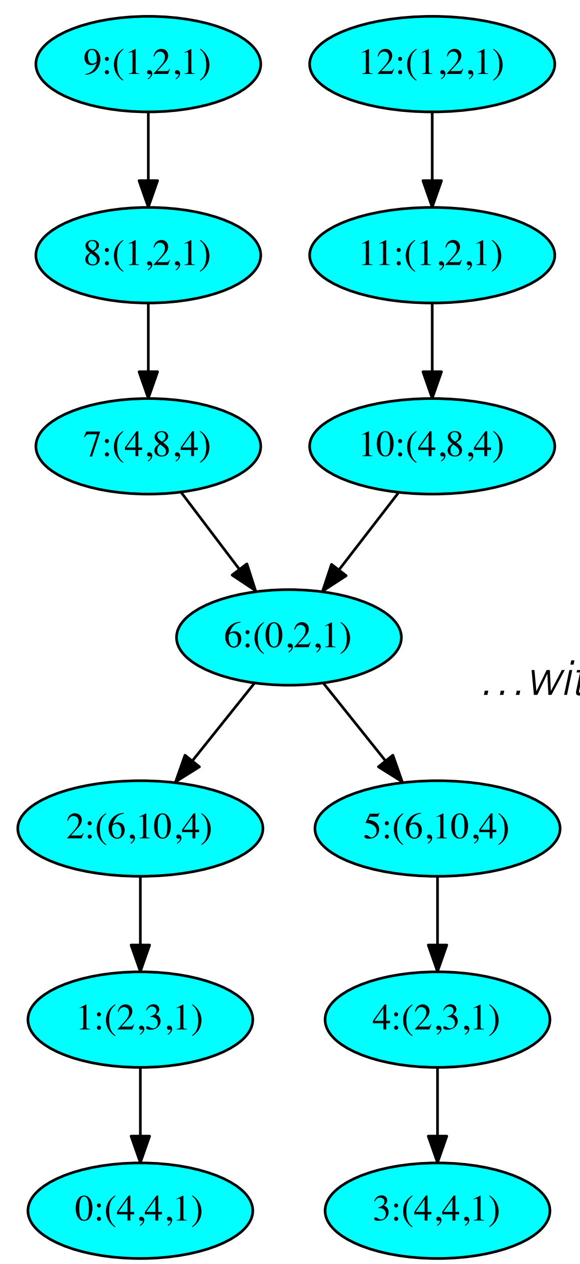


graded chain equivalence

Conley complex

$$\Delta^M = \begin{array}{cc|c|c|c} & & 0 & 3 & 6 \\ & & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 \\ \hline 3 & 0 & 0 & 0 & 1 \\ \hline 6 & 1 & 0 & 0 & 0 \end{array}$$

node index
cell dim.

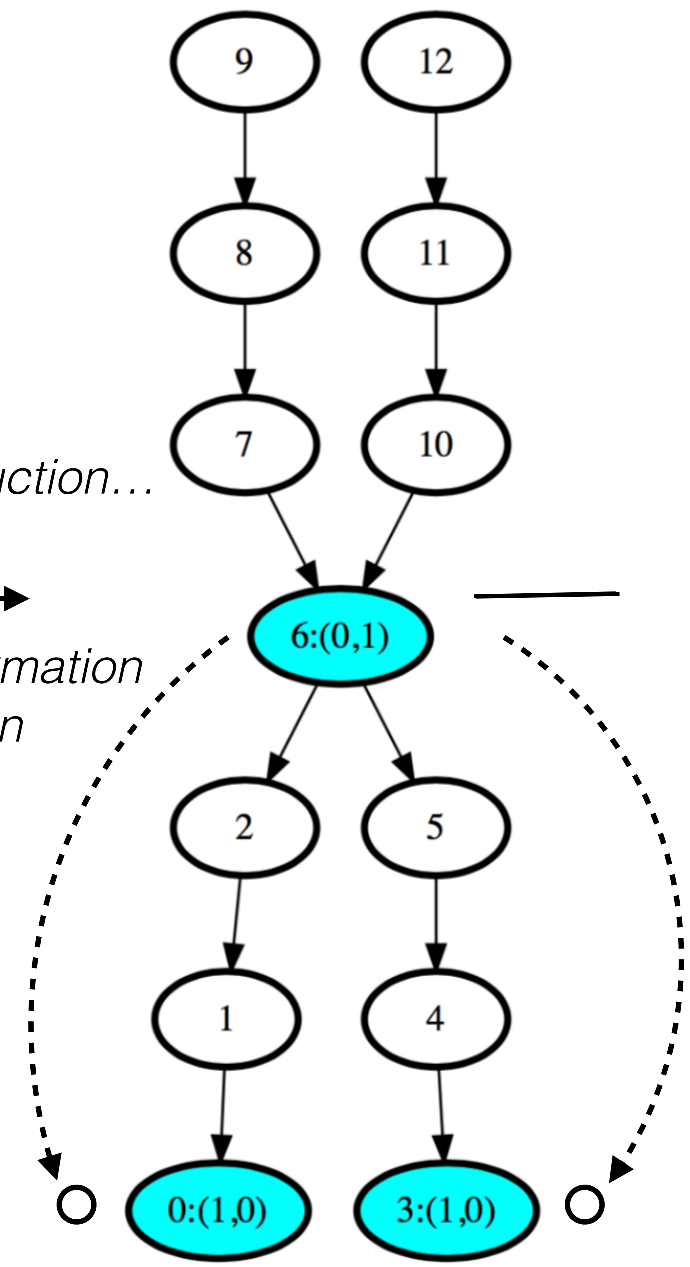


white nodes have no cells (trivial index)

data reduction…

$\xrightarrow{\mathfrak{C}}$

…without information reduction

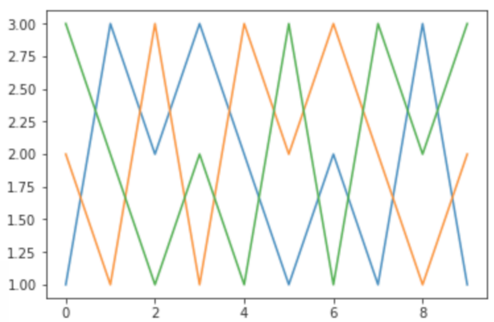


chain-level data compression

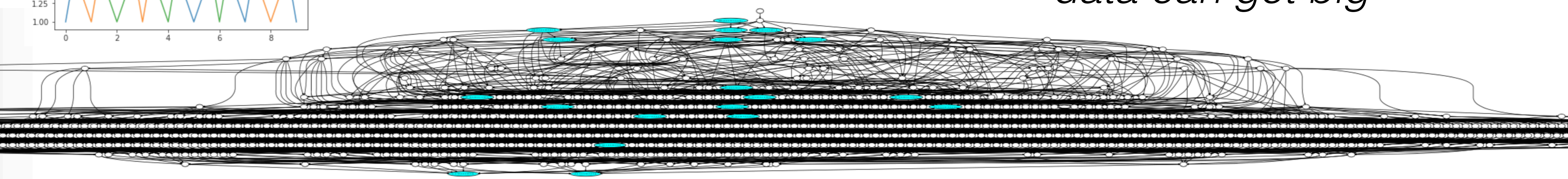
144 cells ⇄ 3 cells

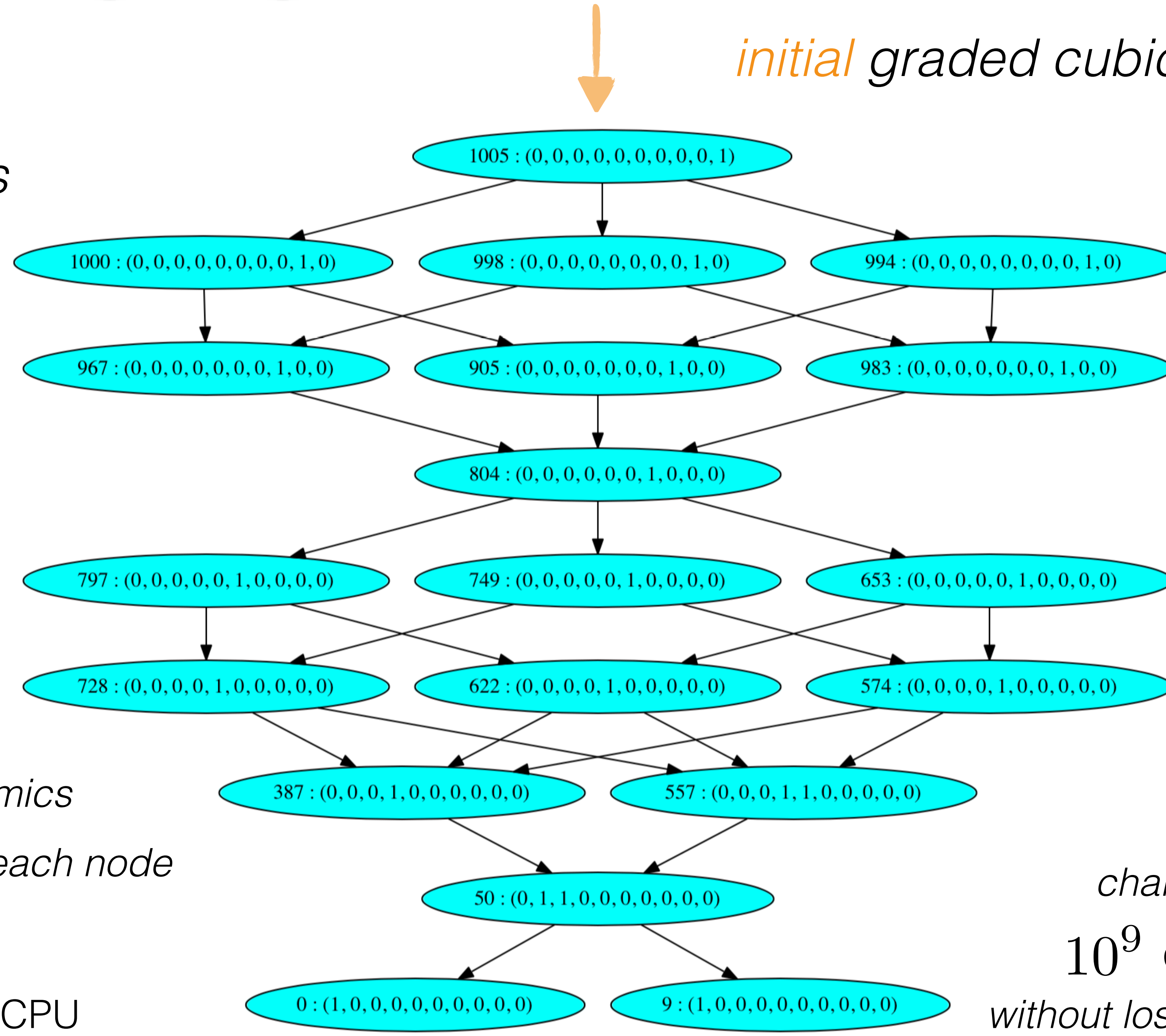
without loss of homological information

braids ii



data can get big





initial graded cubical complex in $\mathbb{R}^9$

10^9 cells

$|\mathsf{SC}(\mathcal{F})| \approx 1000$

restrict poset to nodes with nontrivial index

Conley complex

21 cells

19 nodes

Conley-Morse Graph

organizes global dynamics

Conley index for each node

chain-level data compression

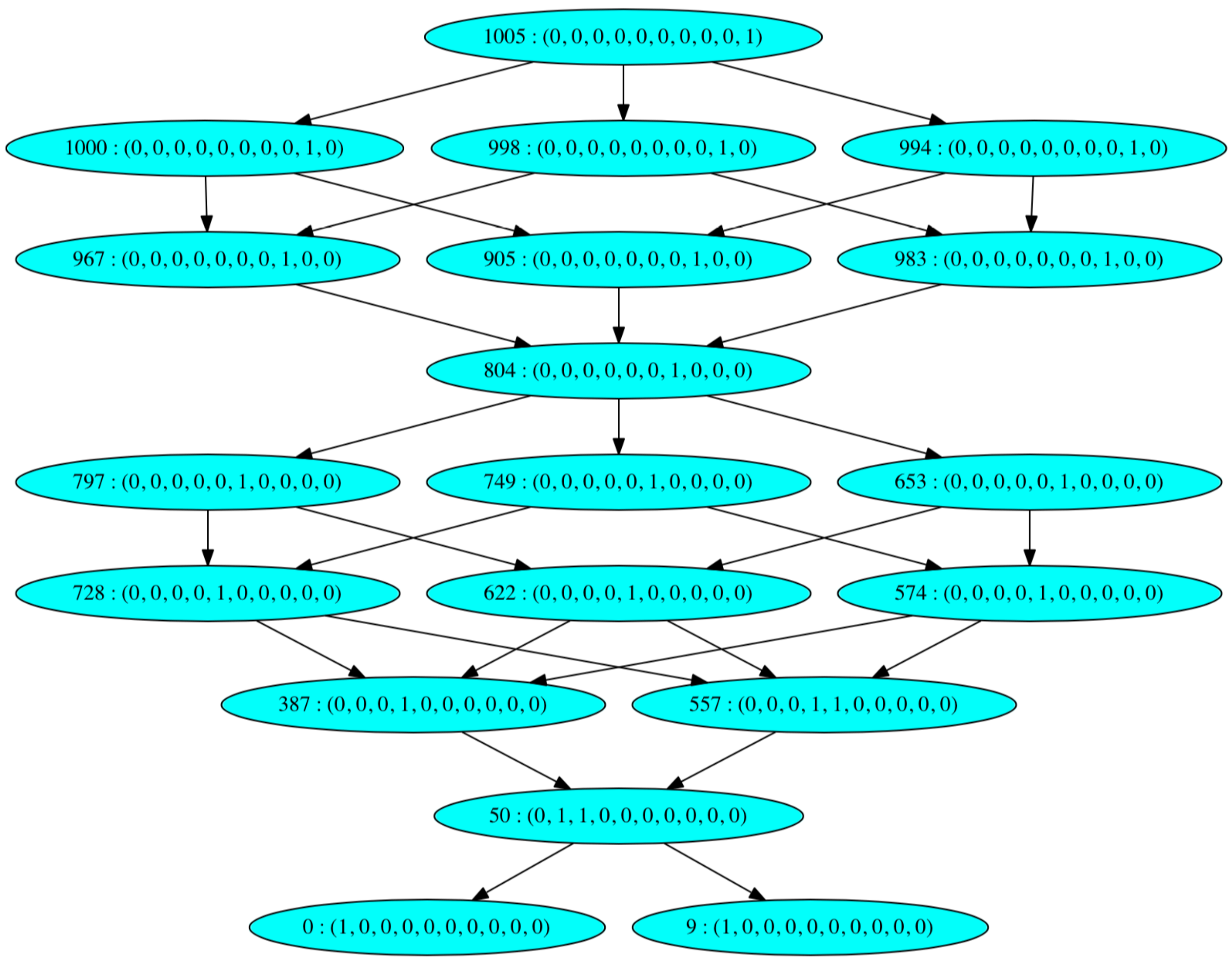
10^9 cells $\rightleftarrows$ 21 cells

without loss of homological information

~47 min on single 2.5 GHz CPU

braids iii

order data



chain data

```
Connection Matrix Data
======================
  Boundaries of 0-cells in Conley complex:
    0 : set()
    1 : set()
  Boundaries of 1-cells in Conley complex:
    2 : {0, 1}
  Boundaries of 2-cells in Conley complex:
    3 : set()
  Boundaries of 3-cells in Conley complex:
    4 : {3}
    5 : {3}
  Boundaries of 4-cells in Conley complex:
    6 : {4, 5}
    7 : {4, 5}
    8 : {4, 5}
    9 : set()
  Boundaries of 5-cells in Conley complex:
    10 : {8, 9, 6}
    11 : {8, 9, 7}
    12 : {9, 6, 7}
  Boundaries of 6-cells in Conley complex:
    13 : set()
  Boundaries of 7-cells in Conley complex:
    14 : {13}
    15 : {13}
    16 : {13}
  Boundaries of 8-cells in Conley complex:
    17 : {14, 15}
    18 : {16, 14}
    19 : {16, 15}
```

Conley-Morse Graph

organizes global dynamics

Conley index for each node

Conley Complex *connection matrix*

boundaries can be queried from the data structure

chain maps to move cycles back and forth

5-fold cover

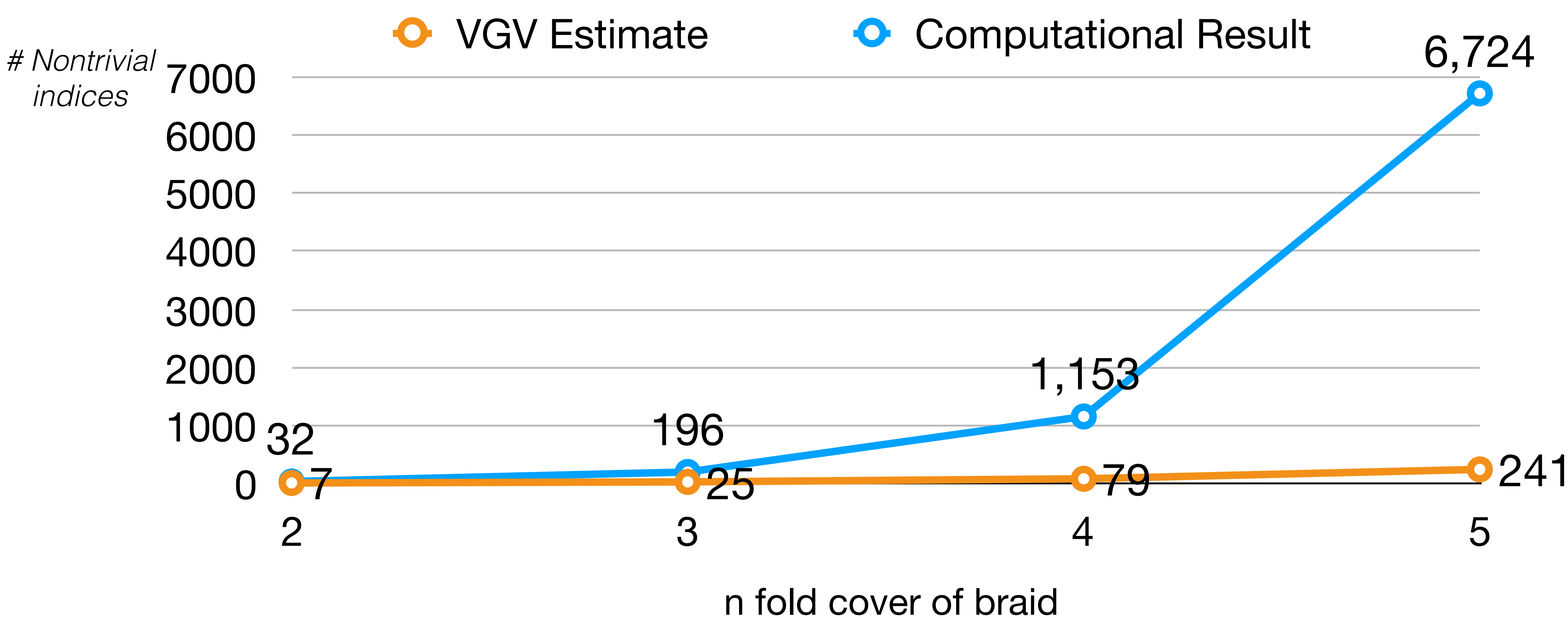
Fig. 7. The lifted skeleton of Example 1 with one free strand

from van den Berg, Ghrist, van der Vorst, Inventiones Math. 2003

Theorem (van den Berg, Ghrist, Vandervorst)

For an n-fold cover of this braid there are at least $3^n - 2$ nontrivial Conley indices

Compare this estimate to our computational result:



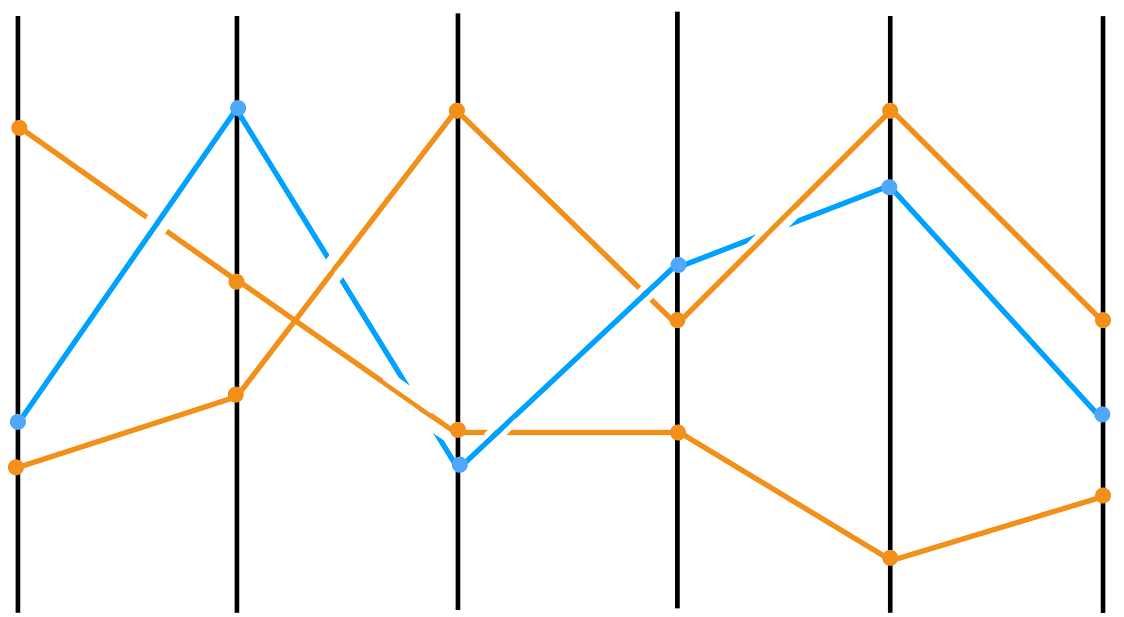
Remark: the 5-fold cover gives a 10-dimensional graded complex containing over 60 billion cells

application iii:

database approach to dynamics

look at all possible five dimensional braids on three strands

i.e. all braids of this type (positive crossings)



$$S_3 \times S_3 \times S_3 \times S_3 \times S_3$$

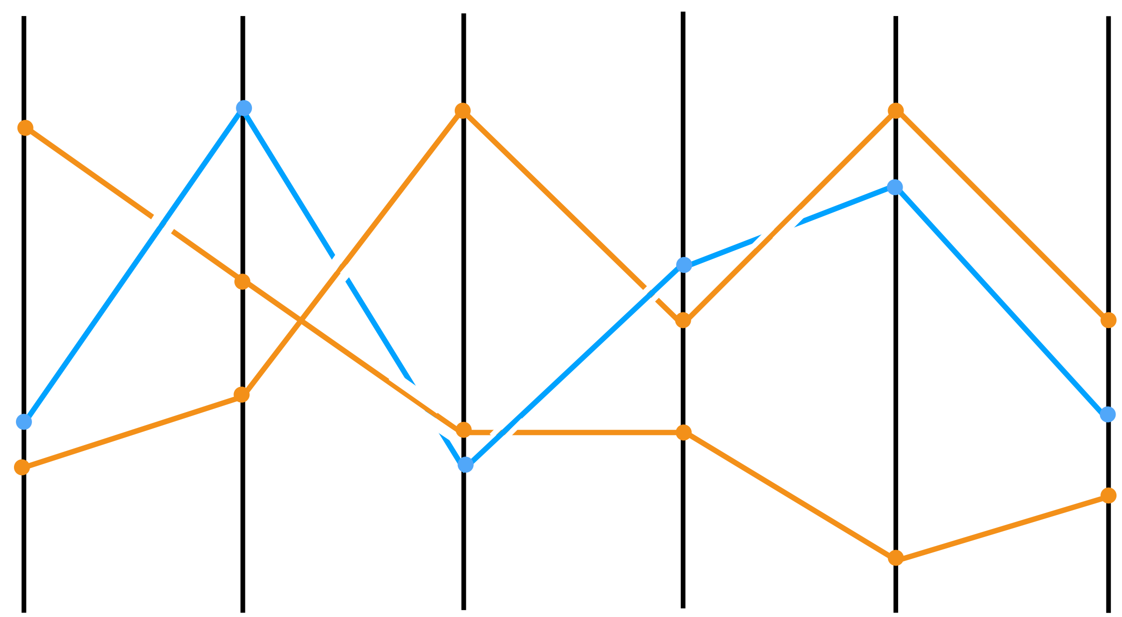
compute database of all Conley complexes

each braid gives a graded cubical complex with 100,000 cells

$$|S_3 \times S_3 \times S_3 \times S_3 \times S_3| = 7776$$

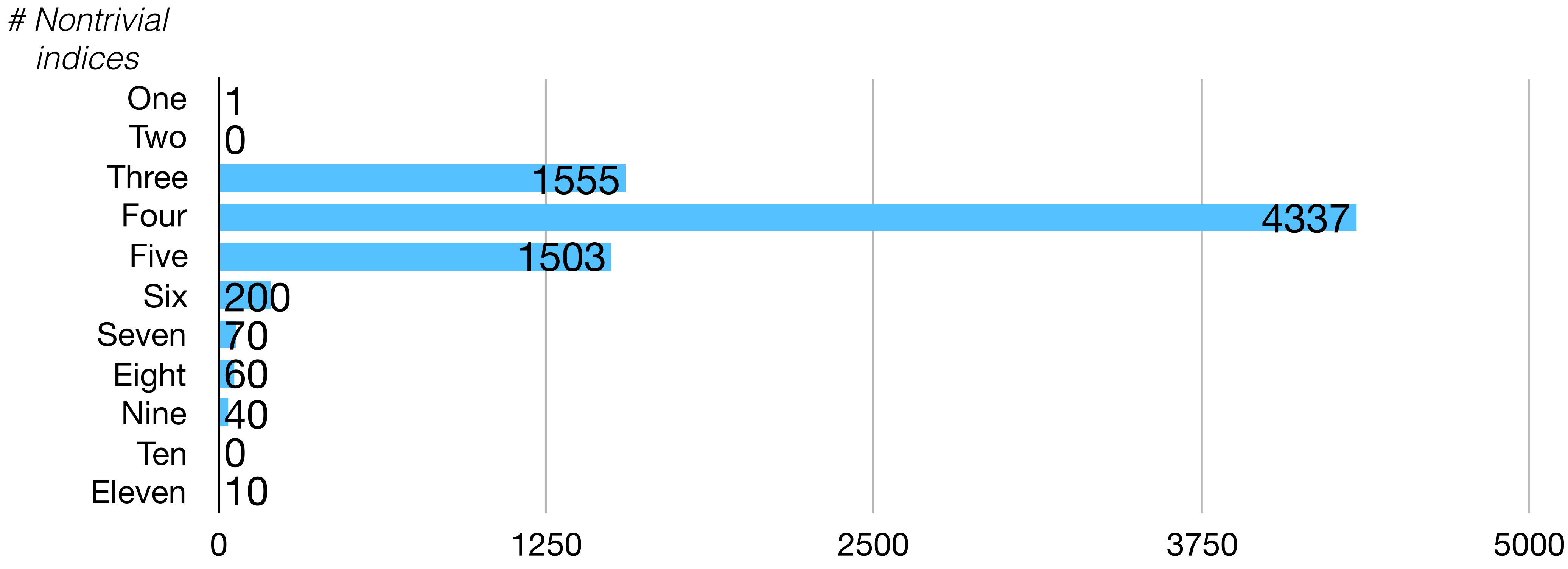
database of 7776 Conley complexes

database of all Conley complexes for 5-D braids on 3 strands



query the database:

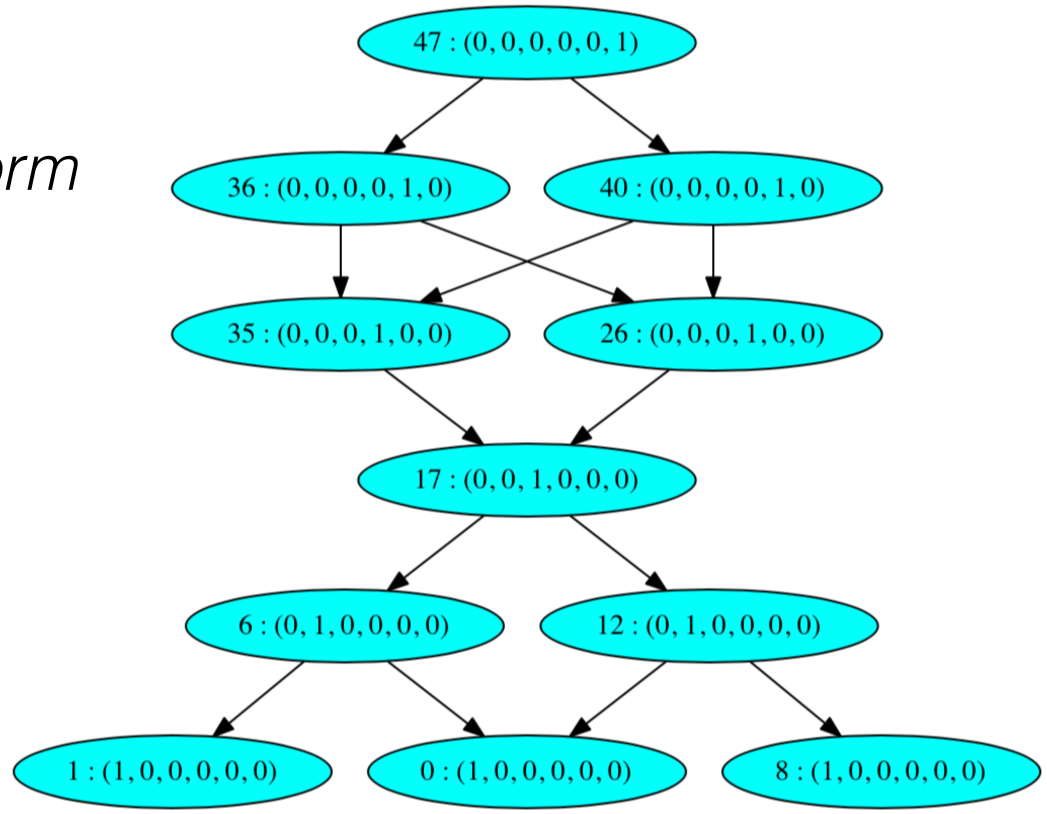
'how many braids have precisely n nontrivial Conley indices?'



query the database:

'what are the Conley-Morse graphs that have 11 nontrivial indices?'

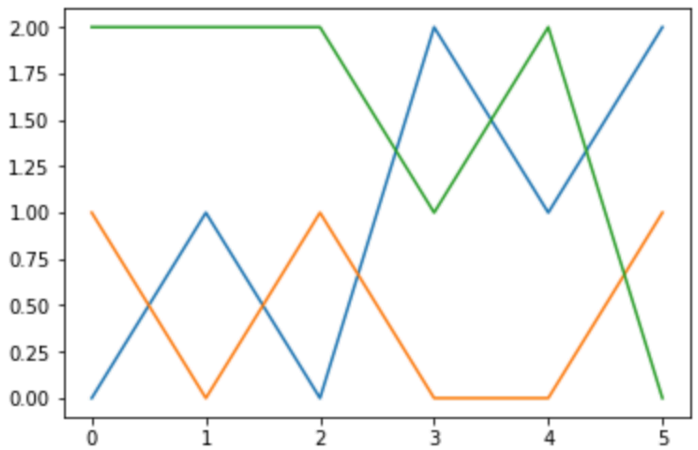
all 10 are of this form

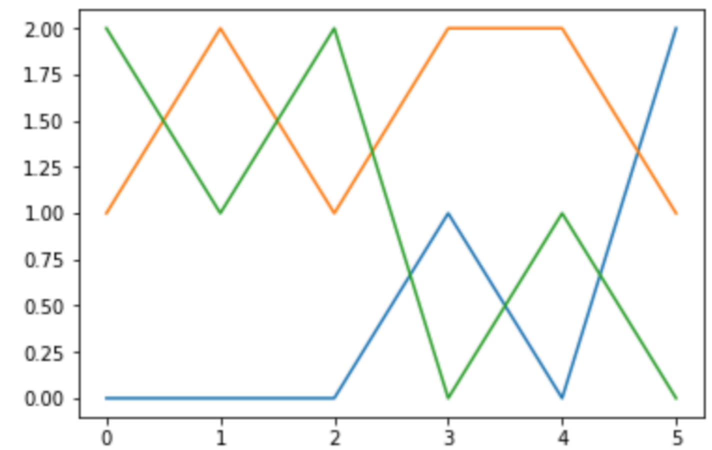


query the database:

'what are the braids that produce 11 nontrivial indices?'

the braids are translates of one of these two (dual) braids



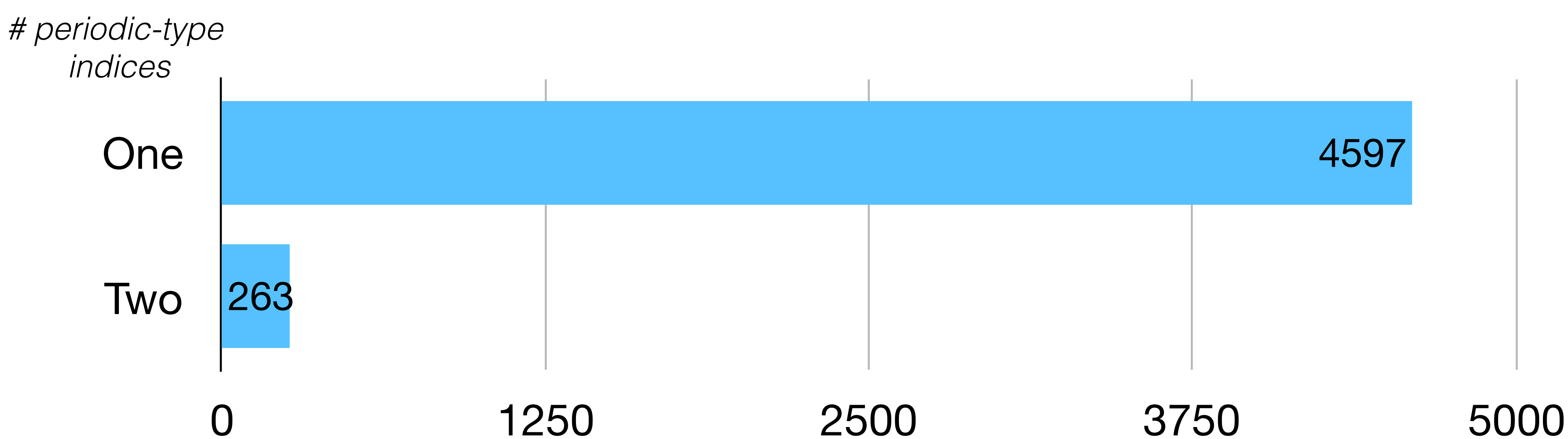


query the database:

‘how many braids produce a Conley index that looks like a periodic orbit?’

i.e. contain one or more of the following indices

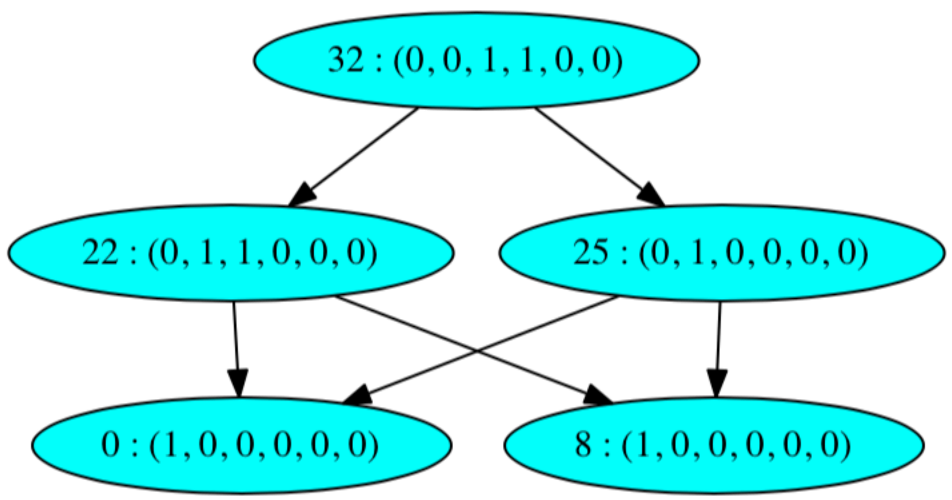
$(1, 1, 0, 0, 0, 0)$ $(0, 1, 1, 0, 0, 0)$ $(0, 0, 1, 1, 0, 0)$ $(0, 0, 0, 0, 1, 1)$

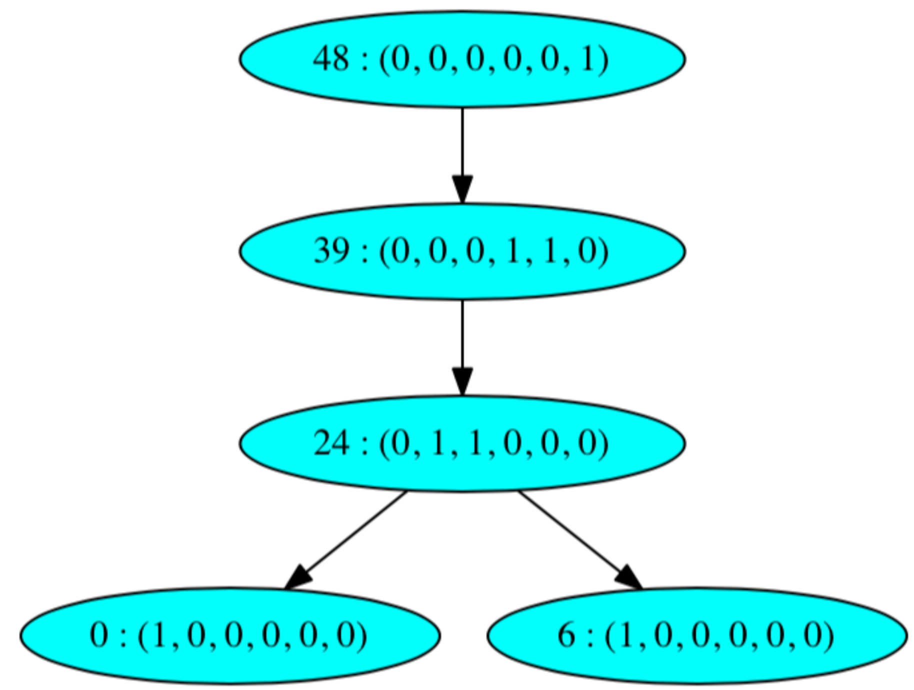


query the database:

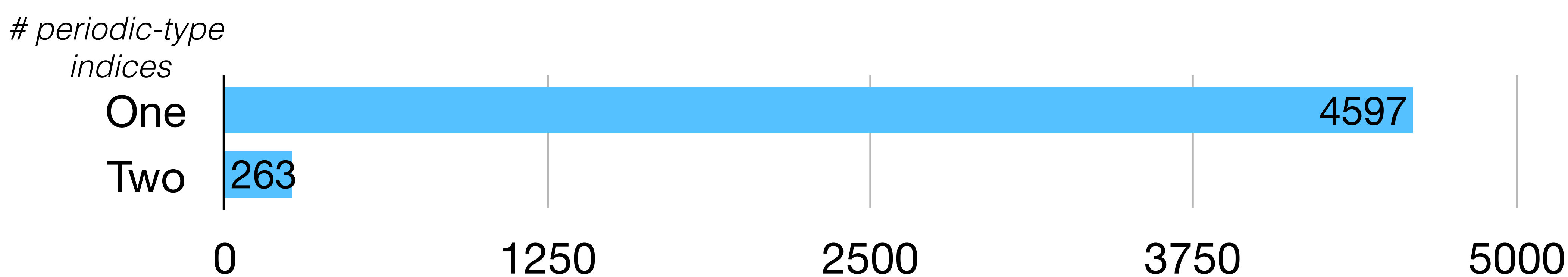
‘what are the Conley-Morse graphs two or more periodic-type indices?’

all of the 263 are of the following two forms:





upshot: we can examine not only high-dimensional braids,
but also ask questions about dynamics in the space of braids



thank you for your attention

Collaborators:
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K. Mischaikow
R. van der Vorst