# toward a computational homological theory of dynamics

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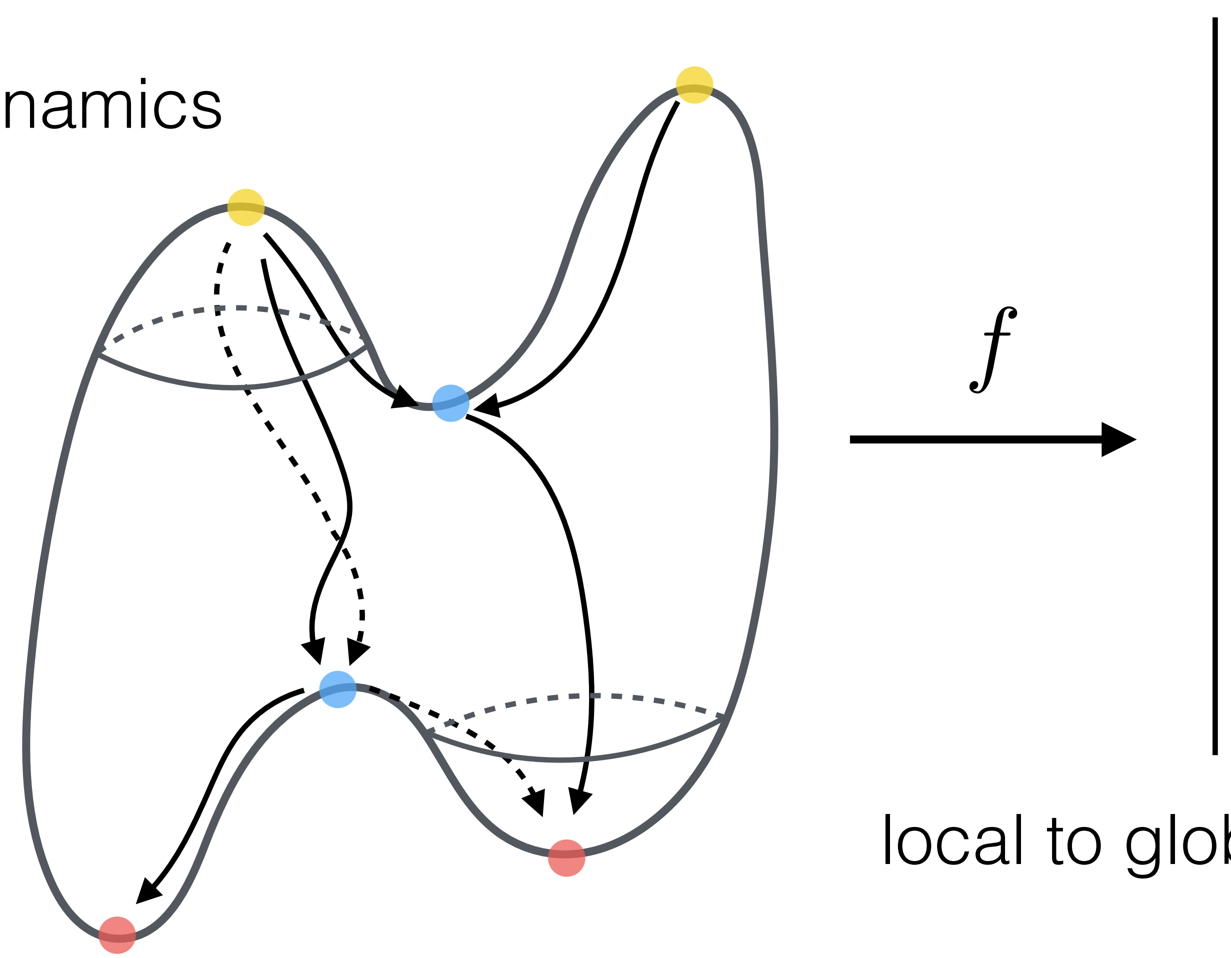
# philosophy

- a dynamical system engenders topological data
- local data (equilibria) and global data (attractors)

data are *ordered* by dynamics

$$\dot{x} = -\nabla f$$




data have *algebraic invariants* (homology)

local to global in algebra

# recipe for a homology theory

- chains: objects to be counted

basis for vector space

- grading: notion of dimension

0-chains,1-chains,2-chains,...

- boundary operator $\partial$

boundary connectivity

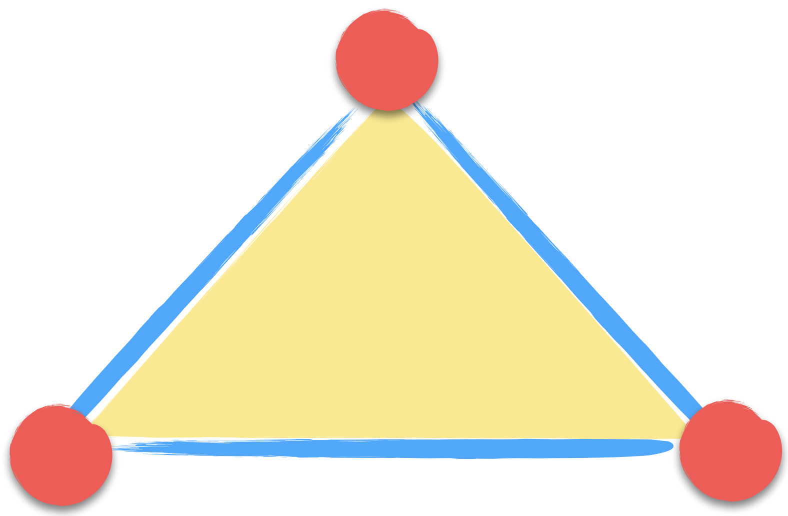
$$\dots \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \dots$$

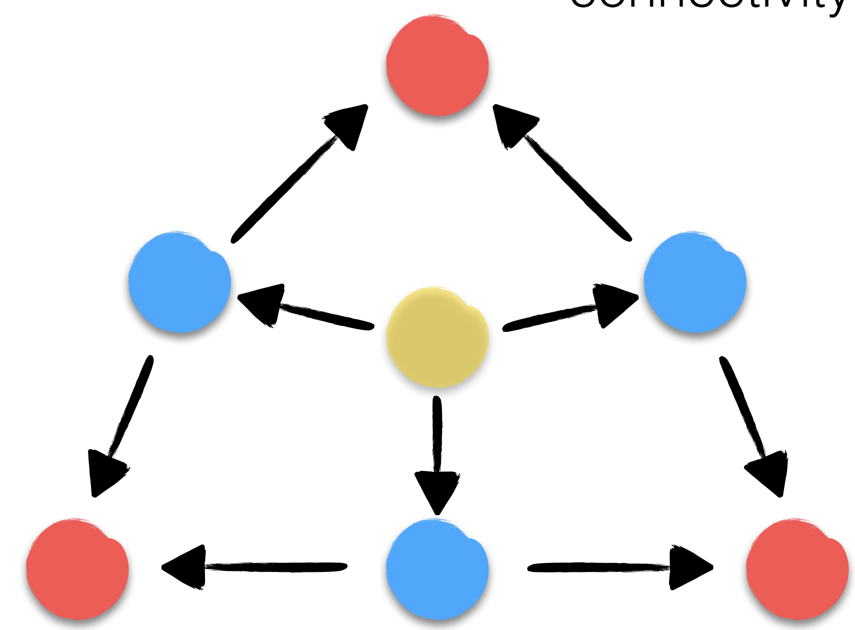
chain complex

boundary cancellation

$$\partial_k \partial_{k+1} = 0$$

# simplicial complex






chains: simplices $\mathbb{Z}_2$

grading: dimension

boundary operator: adjacent simplices

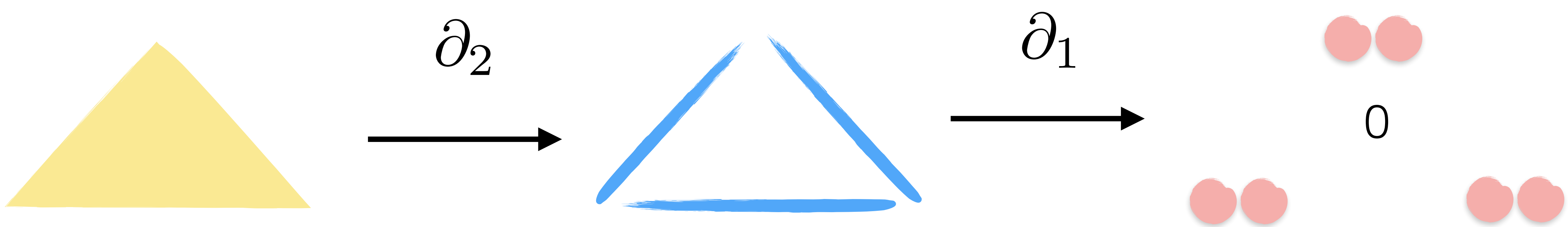
adjacency matrix

$$\partial = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

homology: $\bigoplus_k H_k$ graded vector space

$$H_k := \frac{Ker(\partial_k)}{Im(\partial_{k+1})}$$

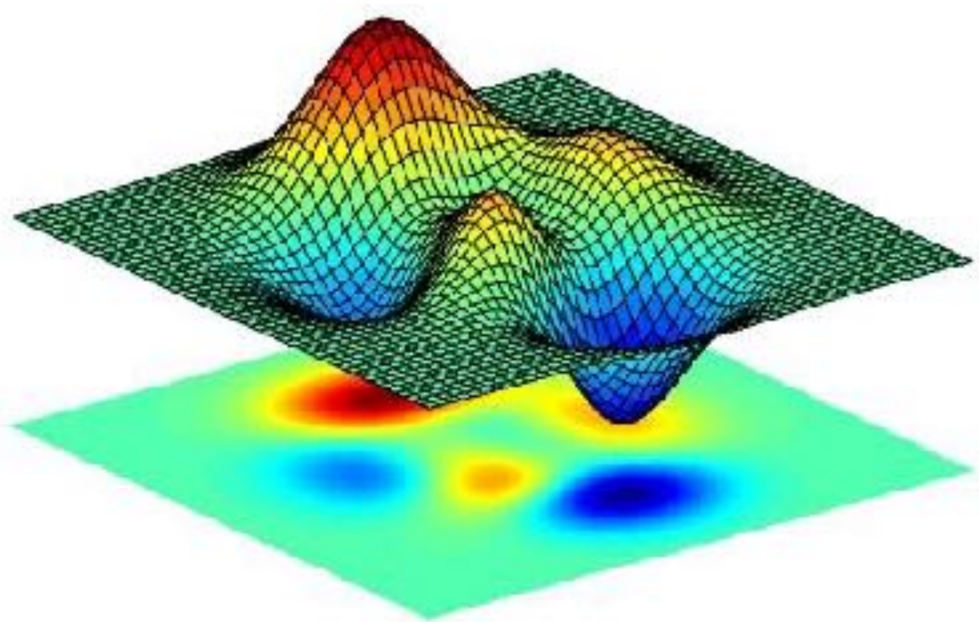
'cycles mod boundaries'




$$H_1 := \frac{Ker\partial_1}{Im\partial_2} = 0$$

# the model of gradient dynamics

$f : X \to \mathbb{R}$



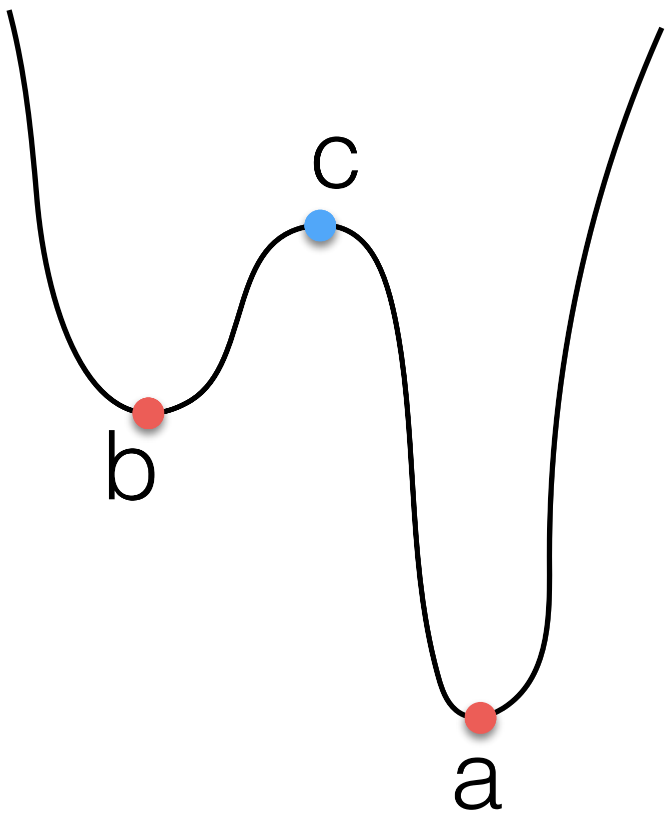
$\dot{x} = -\nabla f(x)$

equilibria, heteroclinic orbits

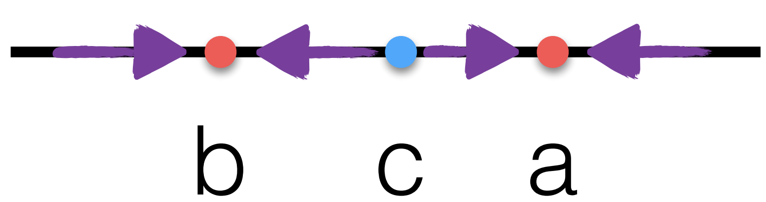
global dynamics → Conley's Decomposition Theorem

(Morse) homology theory for 'nice' $f : \mathcal{M} \to \mathbb{R}$ → Conley-Morse homology

# Morse homology



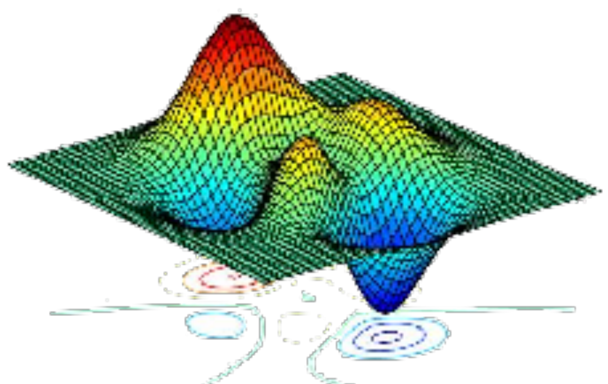

$$\dot{x} = -\nabla f(x)$$

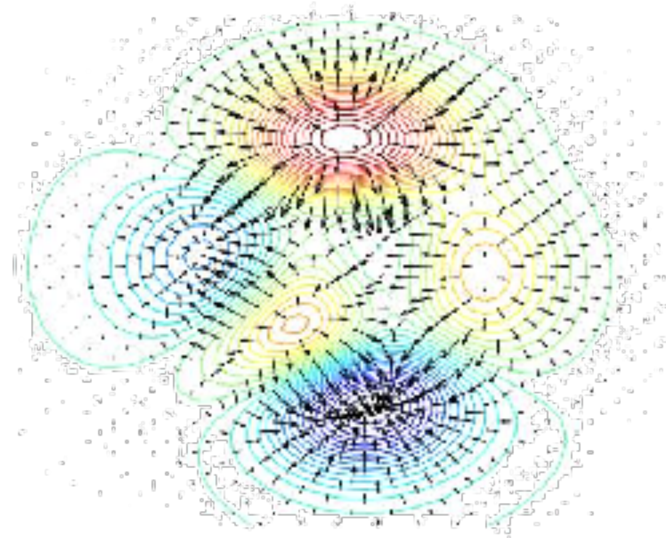


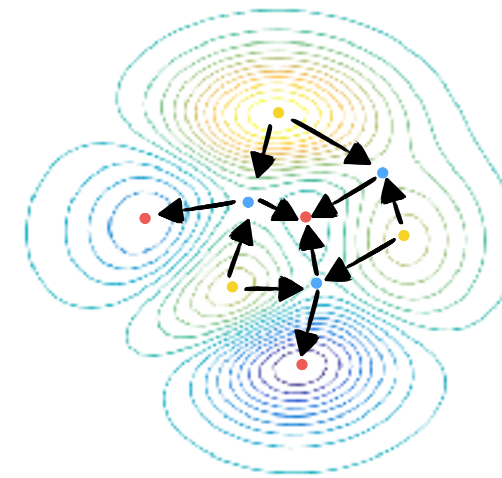

$$\partial_M = \begin{array}{c c} & \begin{array}{ccc} c & b & a \end{array} \\ \begin{array}{c} c \\ b \\ a \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

- chains: non-degenerate fixed points
- grading: number of unstable directions
- (Morse) boundary operator: count the number of connecting orbits (mod 2)

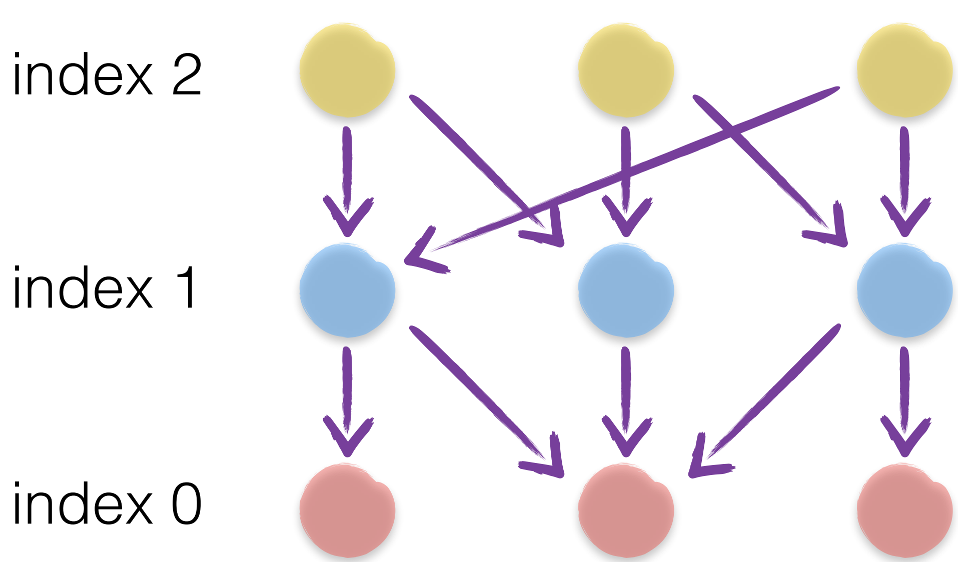
# combinatorial global dynamics







global dynamics are organized by a graph

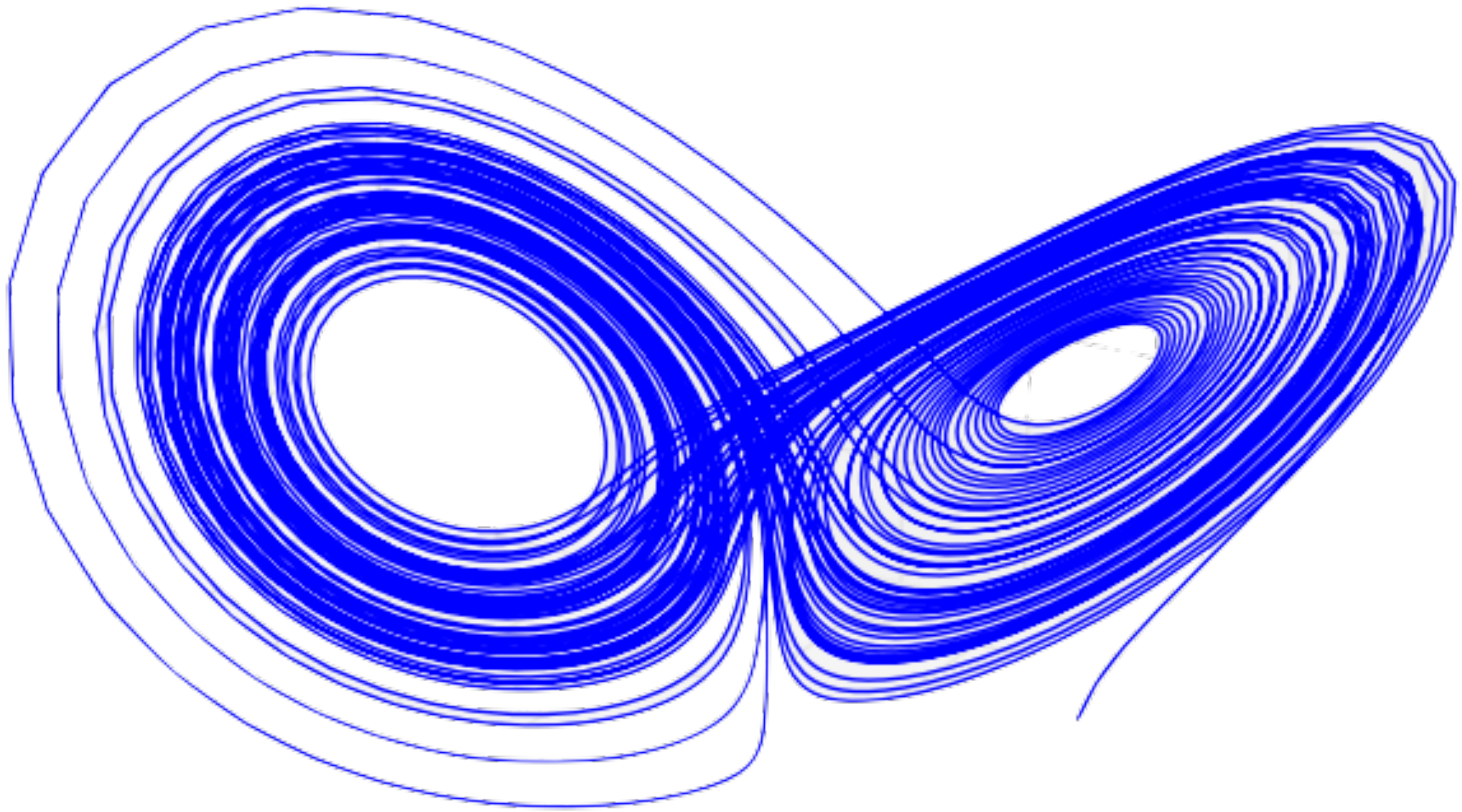



adjacency matrix is the
Morse boundary operator

algebraic representation
of dynamics

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

however…



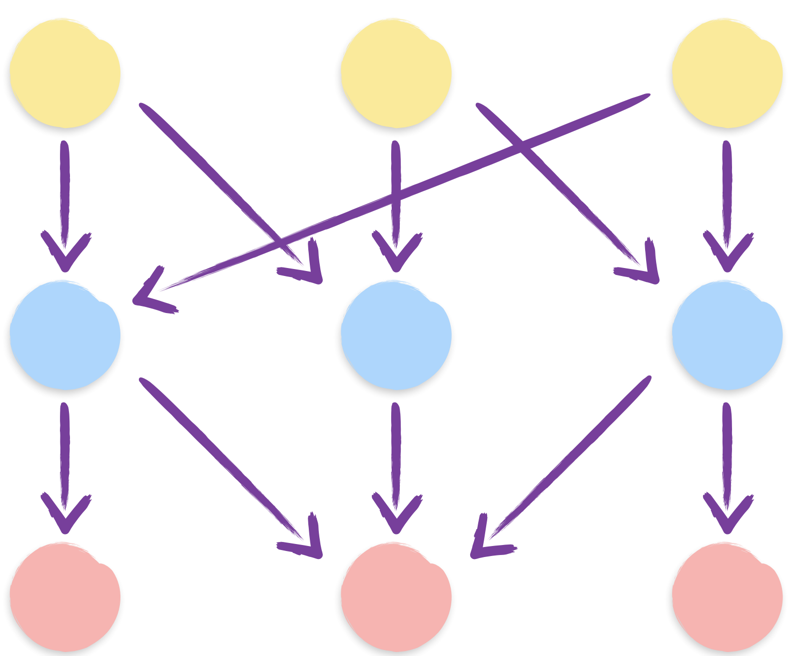
…dynamics can be complex

# neighborhoods instead of invariant sets

a choice of neighborhoods is a choice of…
resolution

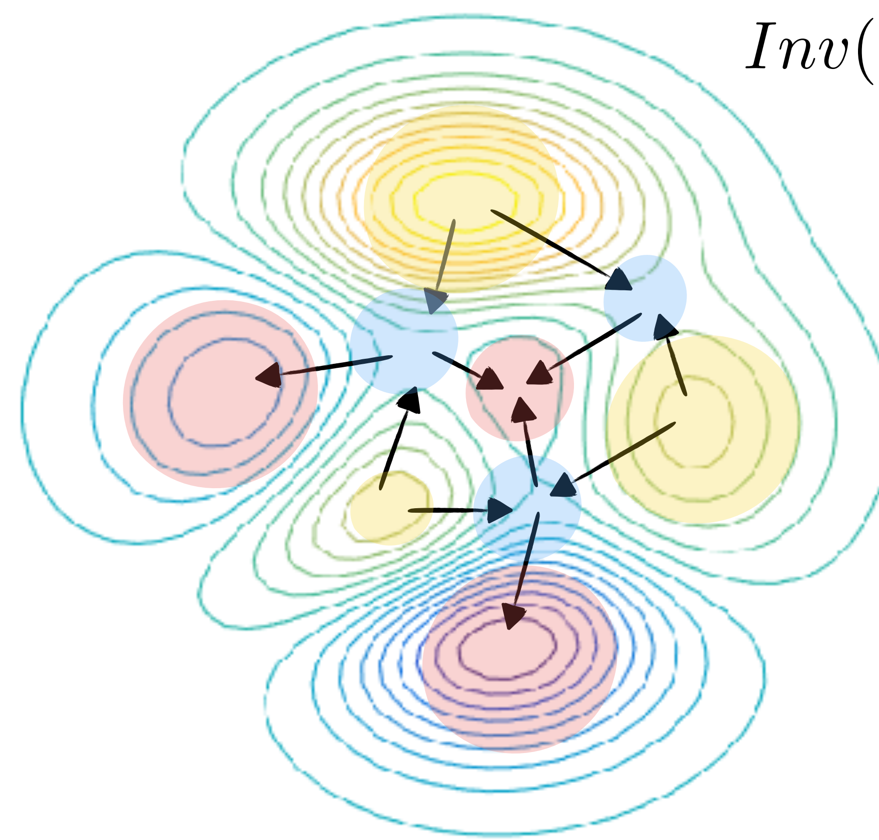
‘Morse decomposition’

C. Conley



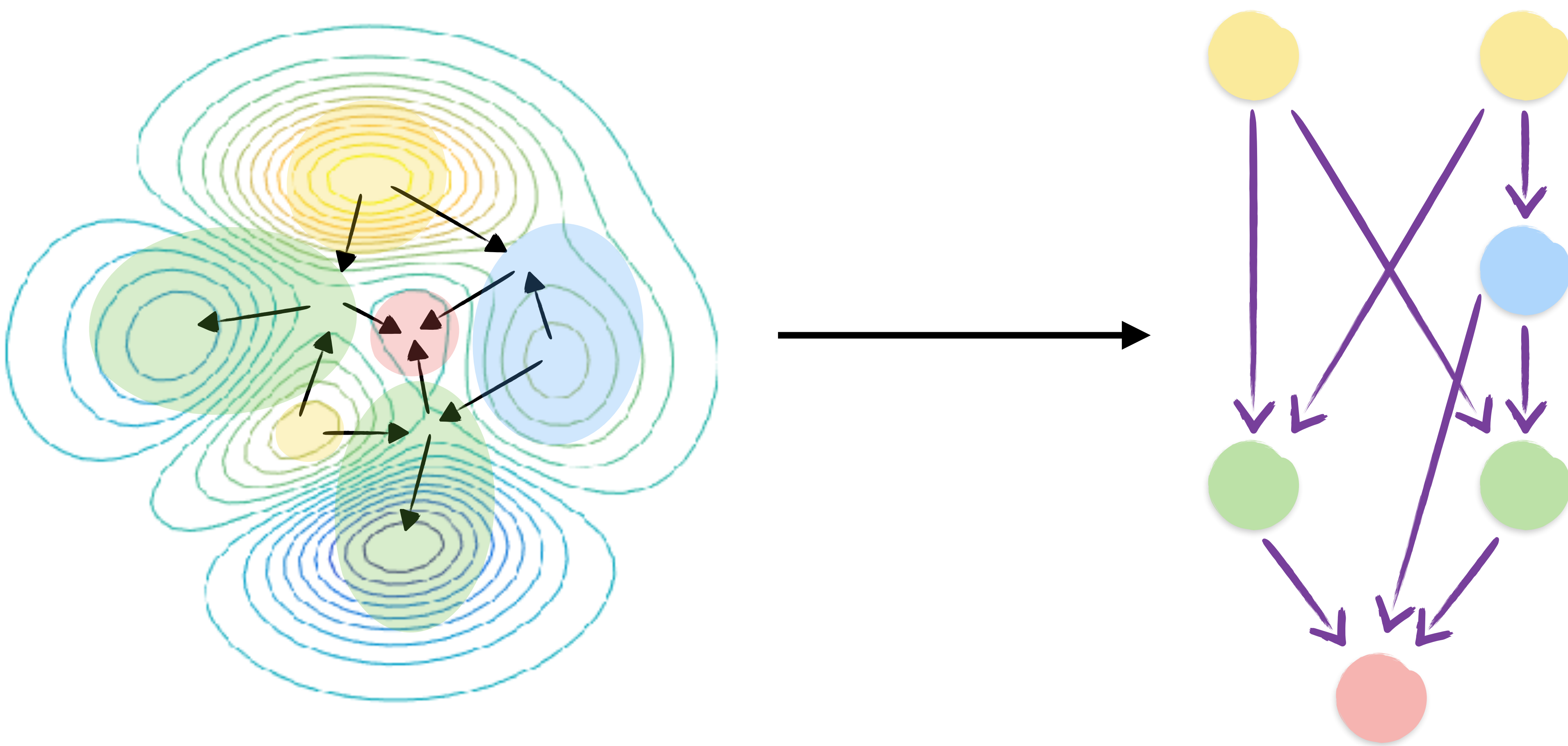
invariant dynamics must be
isolated within neighborhood

$$Inv(N) \subset int(N)$$



at this resolution the decomposition remains the same

# …at a different resolution

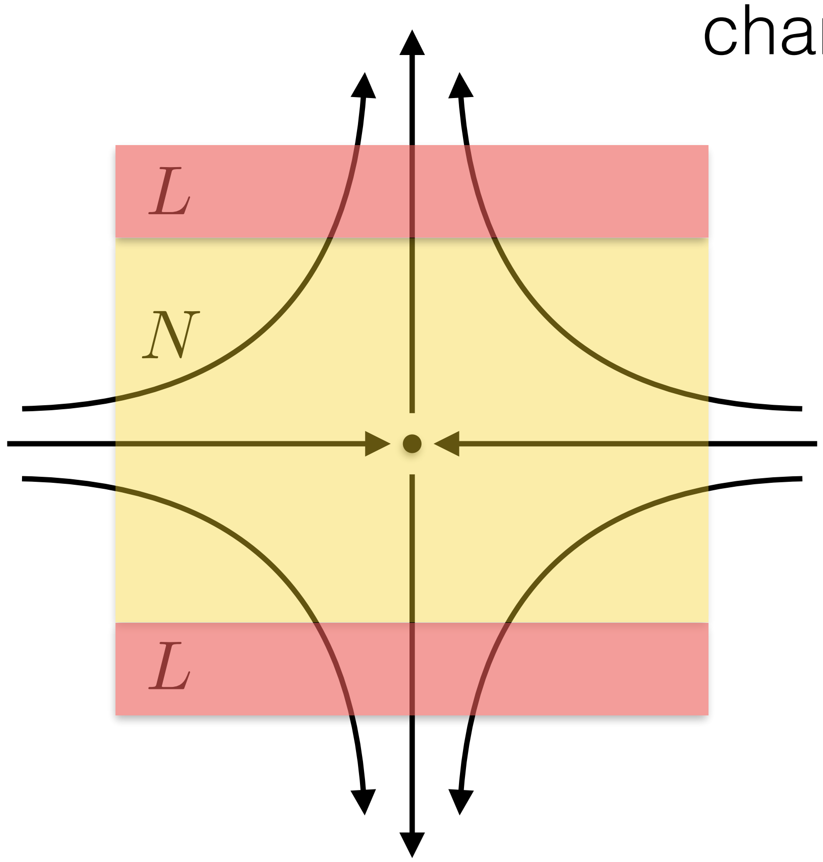


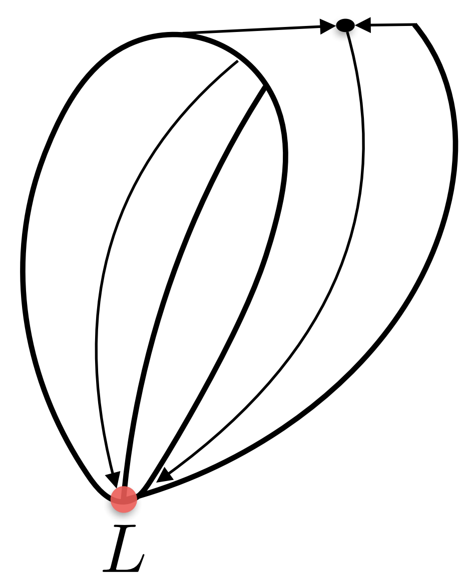
Morse decomposition as a map

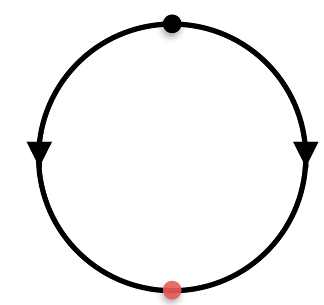
Morse sets as fibers

# Conley index

characterized by dynamics at the boundary



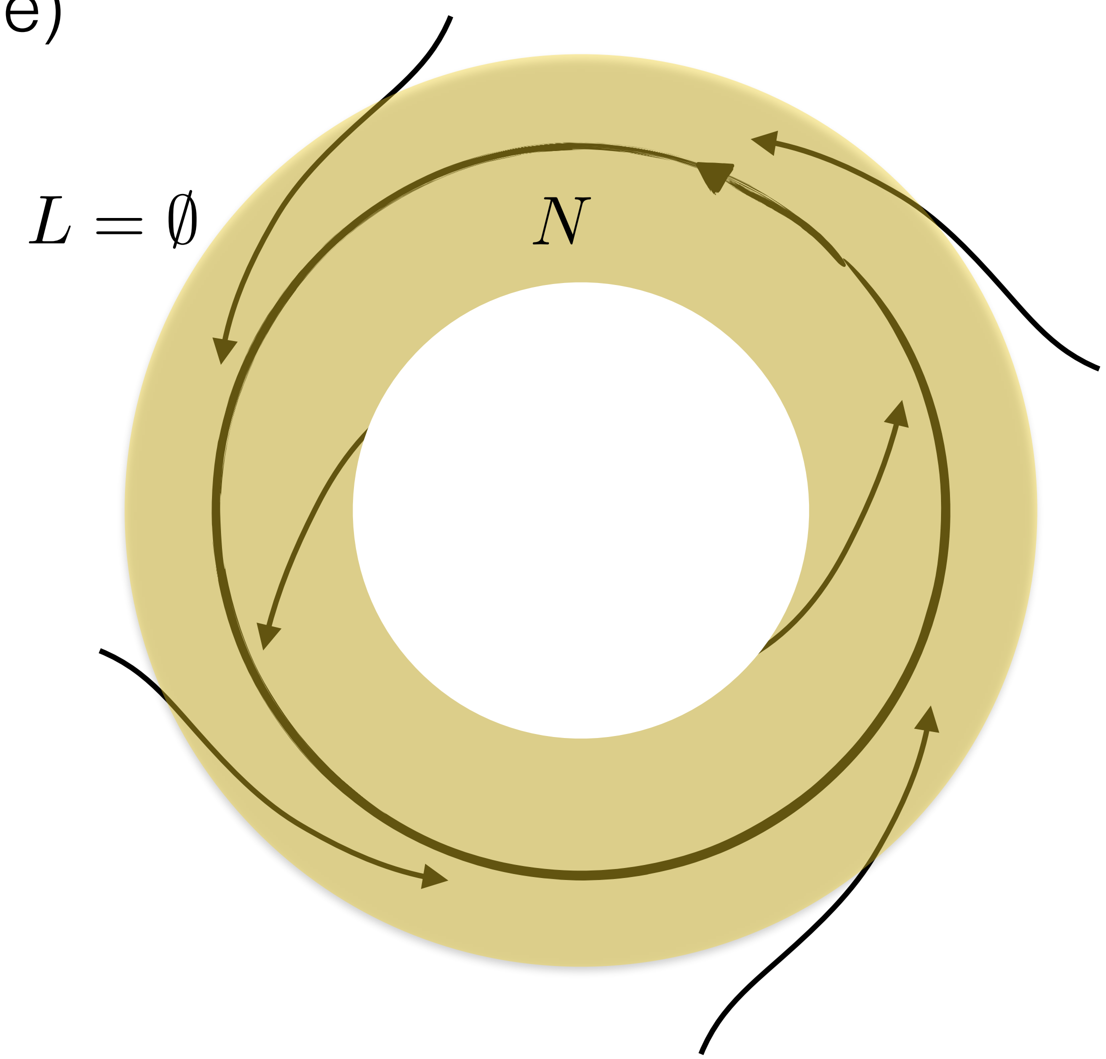
Conley index
(relative homology) $:= \displaystyle\bigoplus_k H_k(N, L) \;=\; 0_0 \oplus \mathbb{F}_1 \oplus 0_2 \oplus \dots$

(caveat: maps requires additional notion of shift equivalence)

# Conley index of stable orbit

$$= \mathbb{F}_0 \oplus \mathbb{F}_1 \oplus 0_2 \oplus \ldots$$

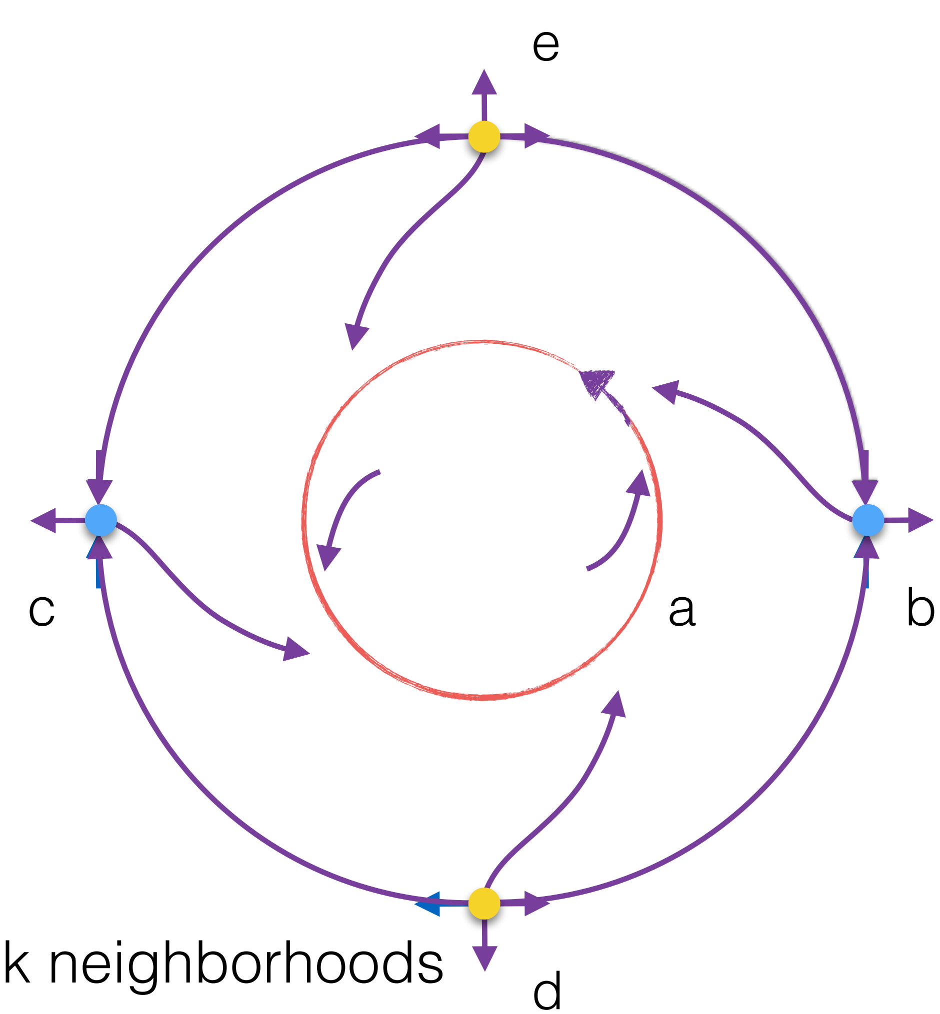
(graded vector space)




## the Conley index…

- is an invariant
- coarsely quantifies unstable dynamics
- has continuation property

# Conley-Morse homology




think neighborhoods

- chains: Conley indices
- grading: come graded
- boundary operator: connection matrix

$$\Delta = \begin{array}{c} \\ \color{gold}{\bullet} \\ \color{gold}{\bullet} \\ \color{cyan}{\bullet} \\ \color{cyan}{\bullet} \\ \circ \\ \color{red}{\bullet} \end{array} \begin{array}{c} \begin{array}{cccc} \color{gold}{\bullet} & \color{gold}{\bullet} & \color{cyan}{\bullet} & \color{cyan}{\bullet} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

not unique

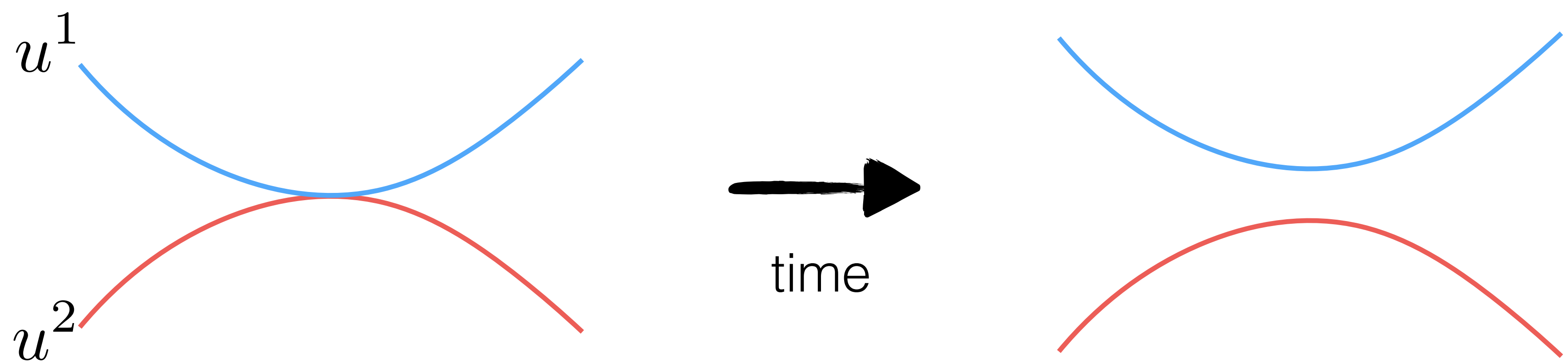
computational Conley theory

combinatorialization

# instantiation: dynamics on braids

$$u_t = u_{xx} + f(x, u, u_x)$$

parabolic dynamics decreases intersections




'comparison principle'

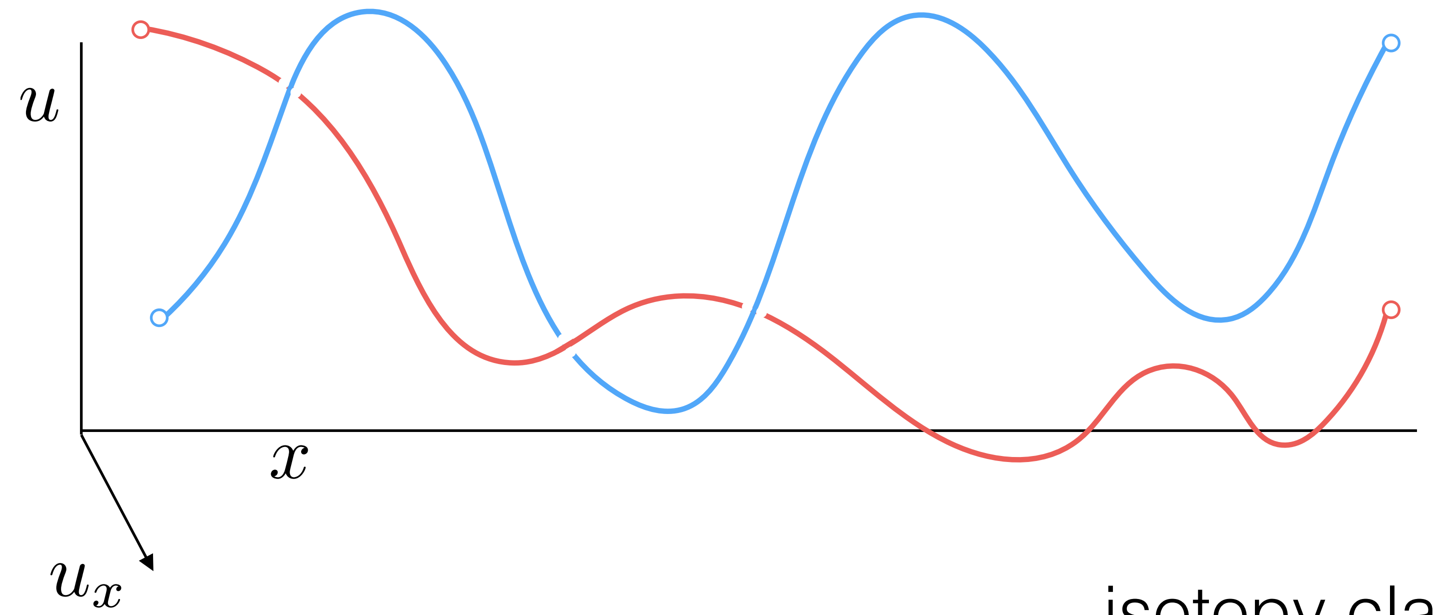
proof:

$$\frac{\partial}{\partial t}(u^1(x,t) - u^2(x,t)) = u^1_{xx} + f(x, u^1, 0) - u^2_{xx} + f(x, u^2, 0)$$

$$= u^1_{xx} - u^2_{xx} > 0$$

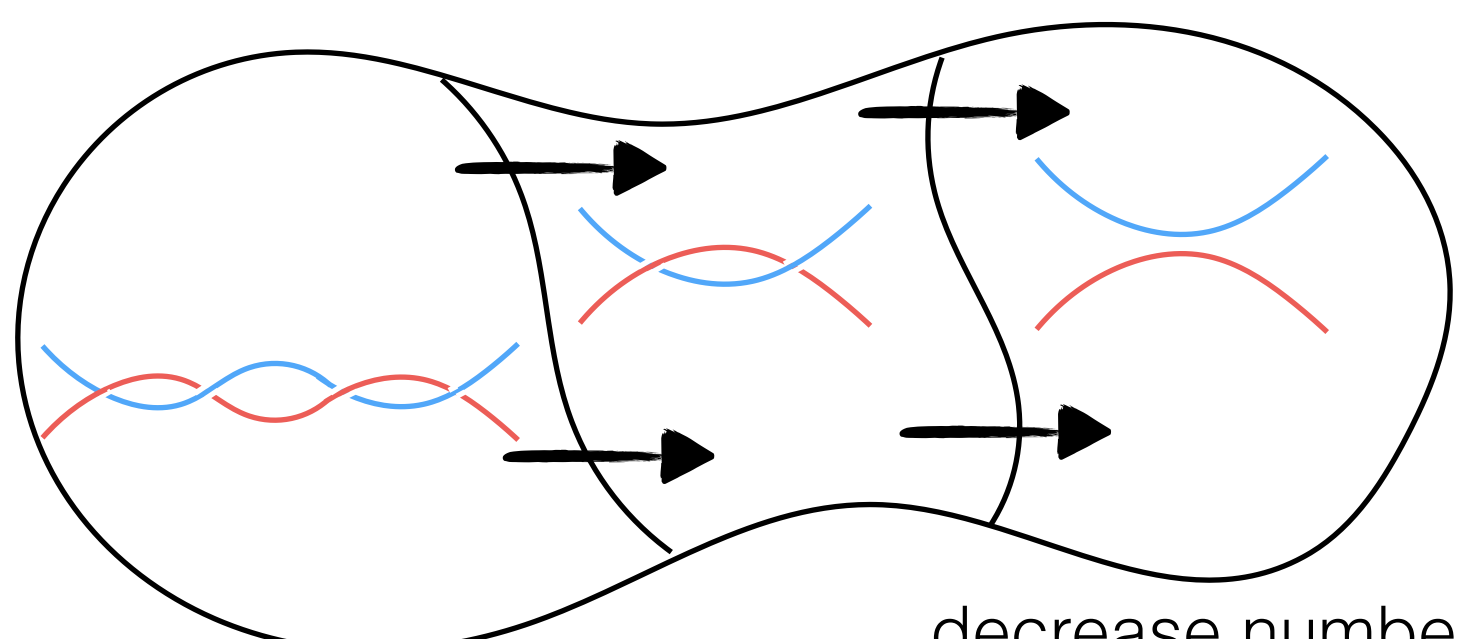
## functions lift to braids

R. Ghrist
R. van der Vorst
J.B. van den Berg




isotopy class, fixed endpoints

## dynamics on braid classes



decrease number of intersections

# functions lift to braids

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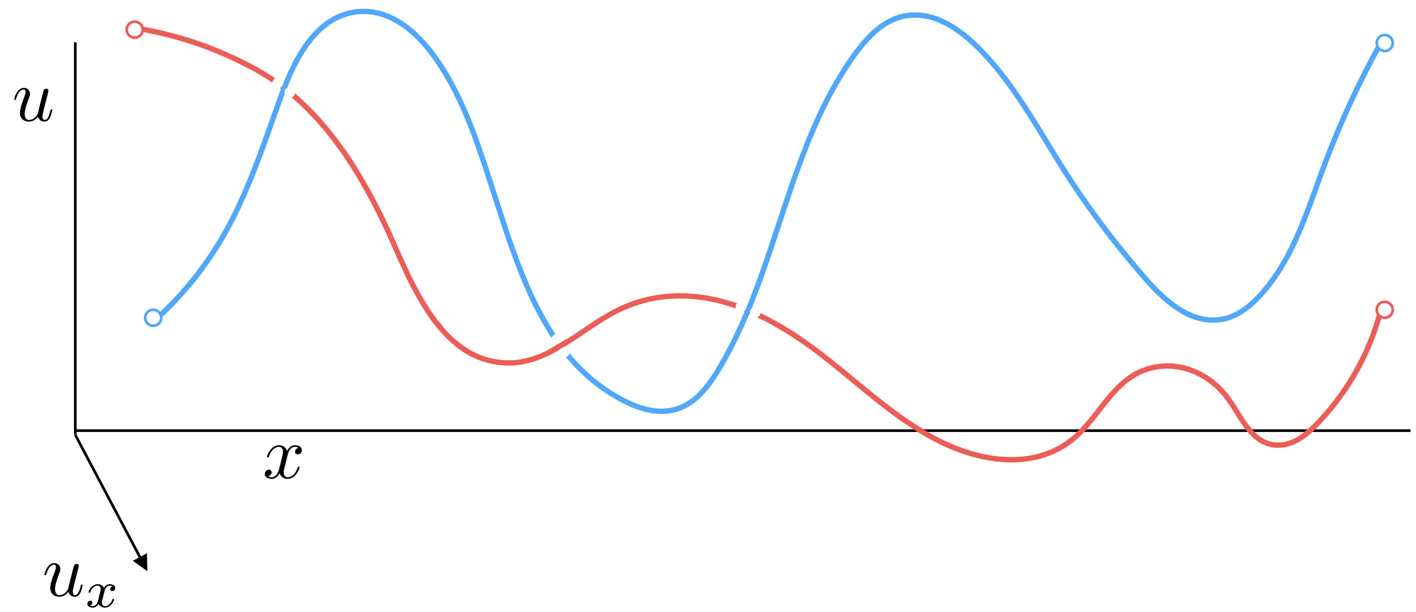
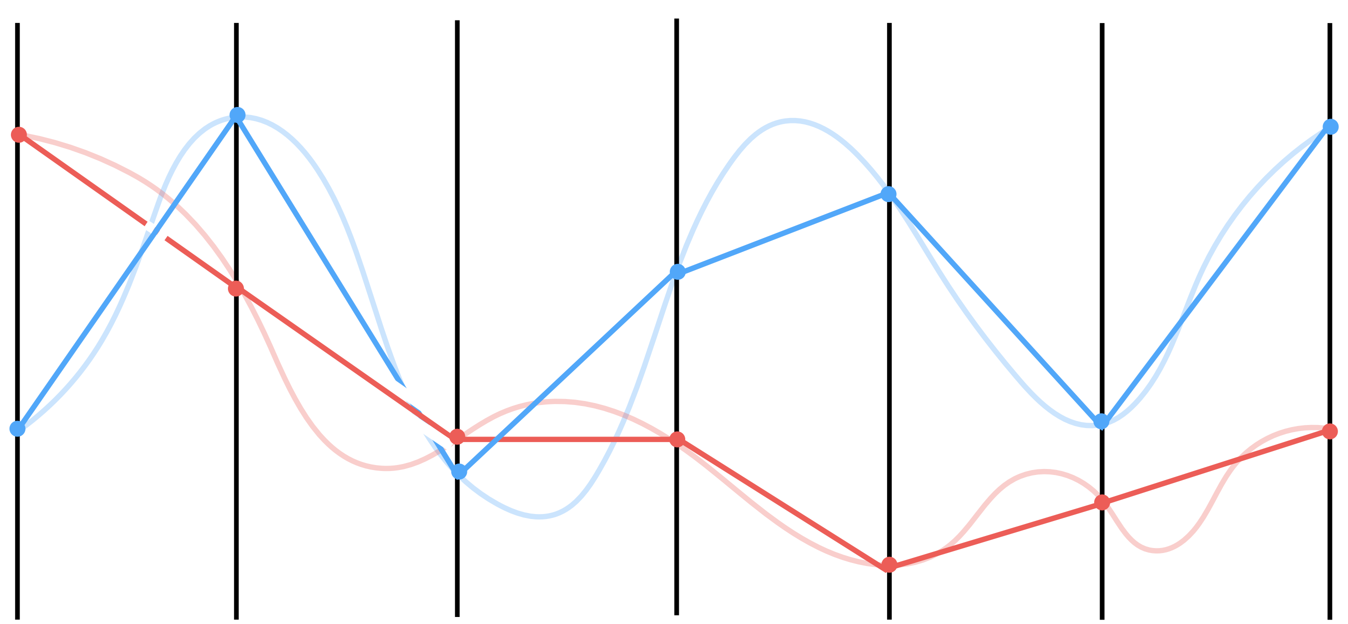
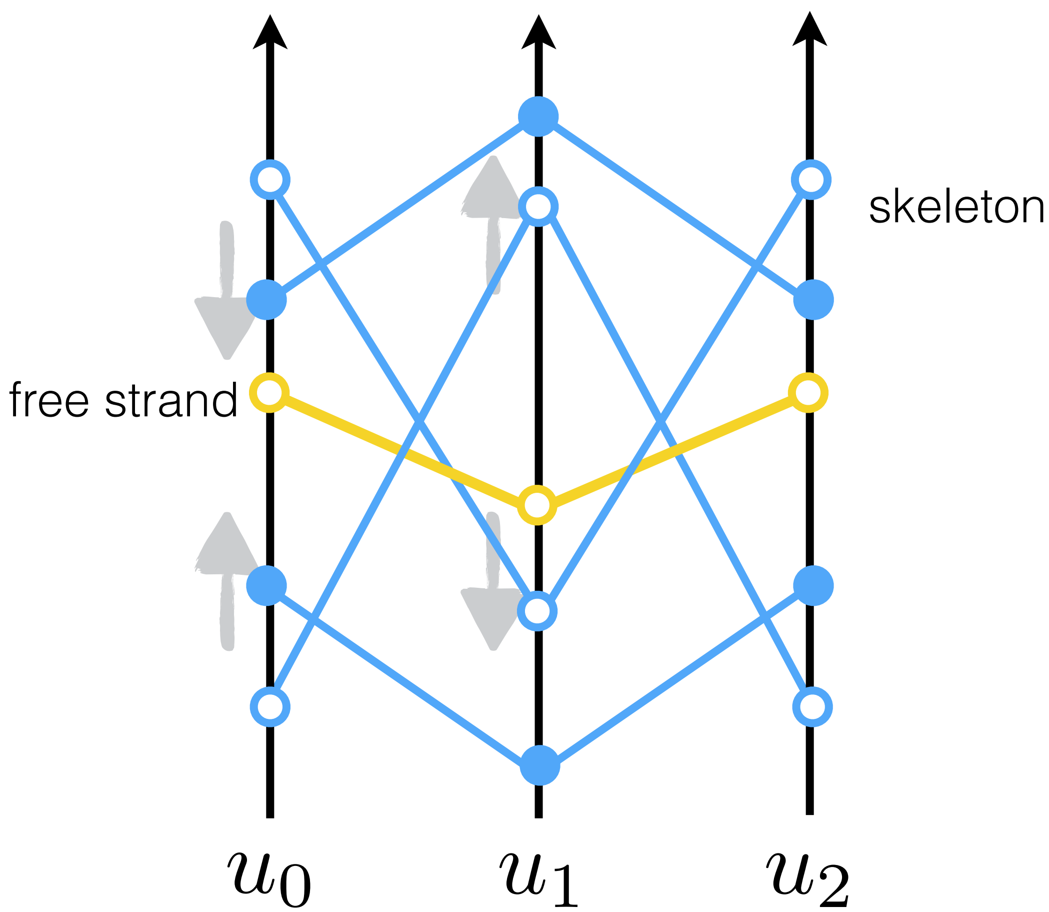



combinatorialization

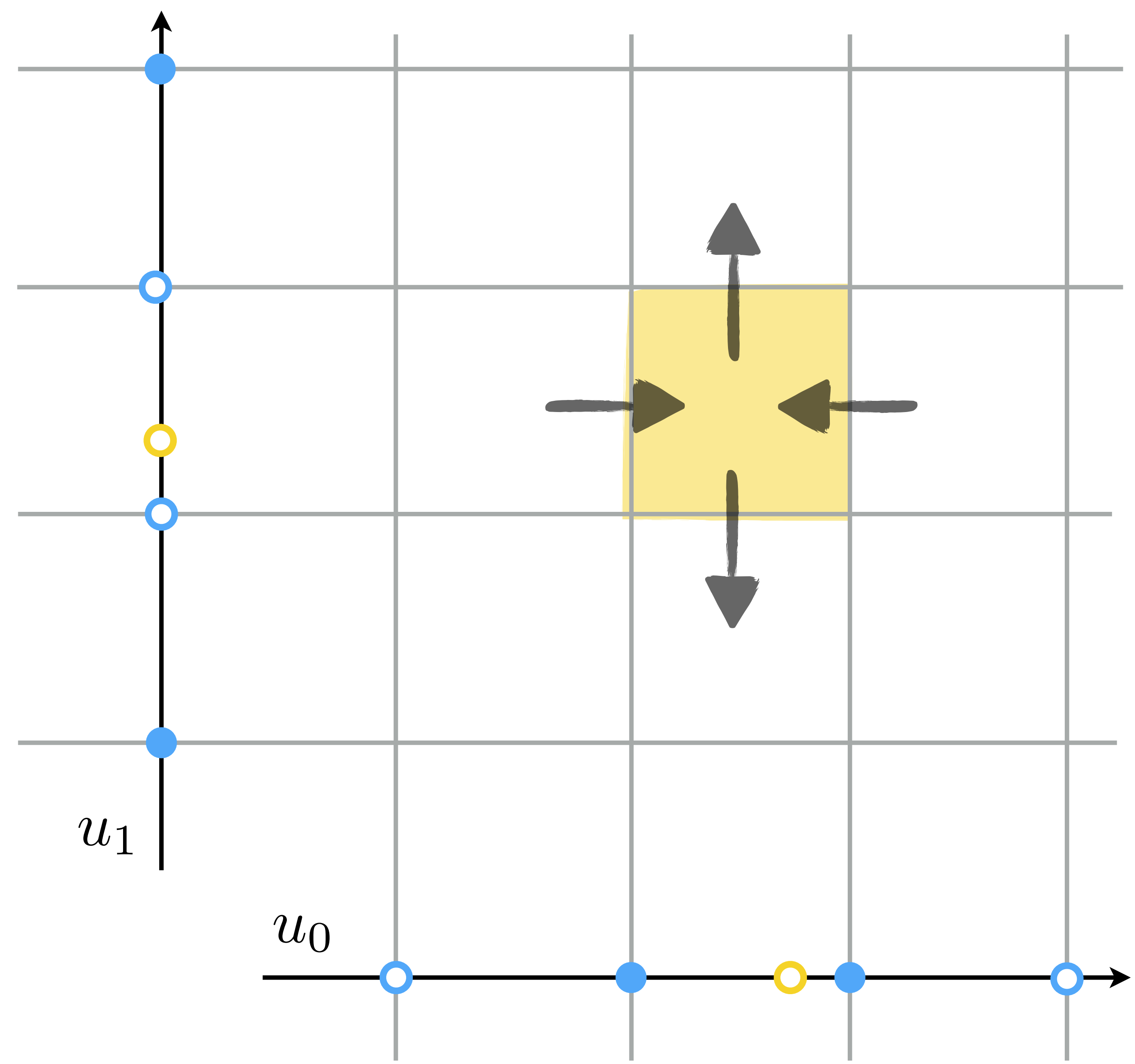





periodic strands

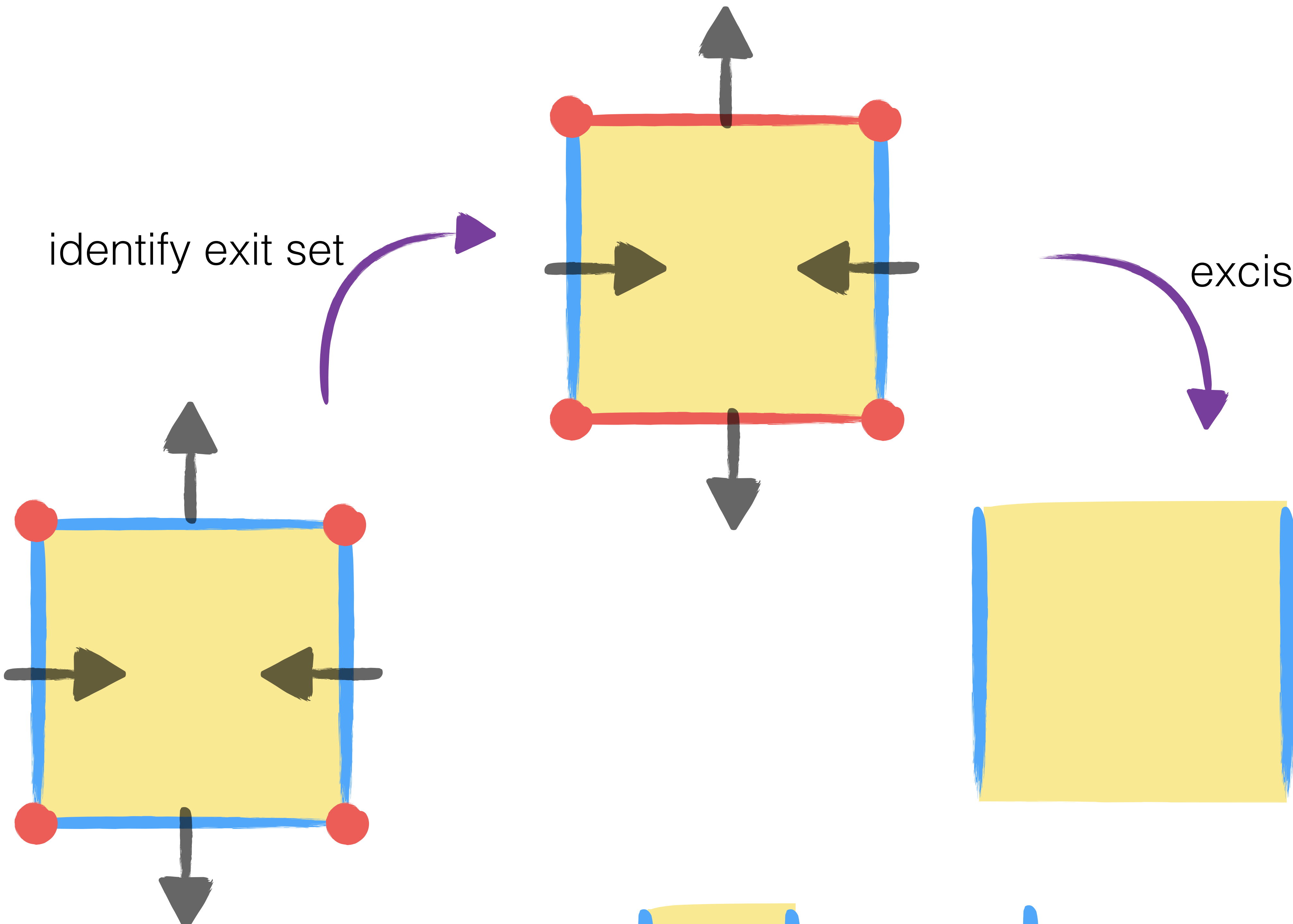
(prevent strand collapse)

## skeleton of stationary solutions

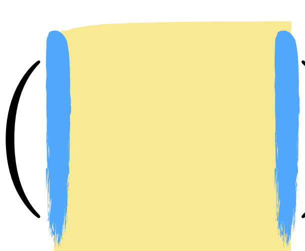


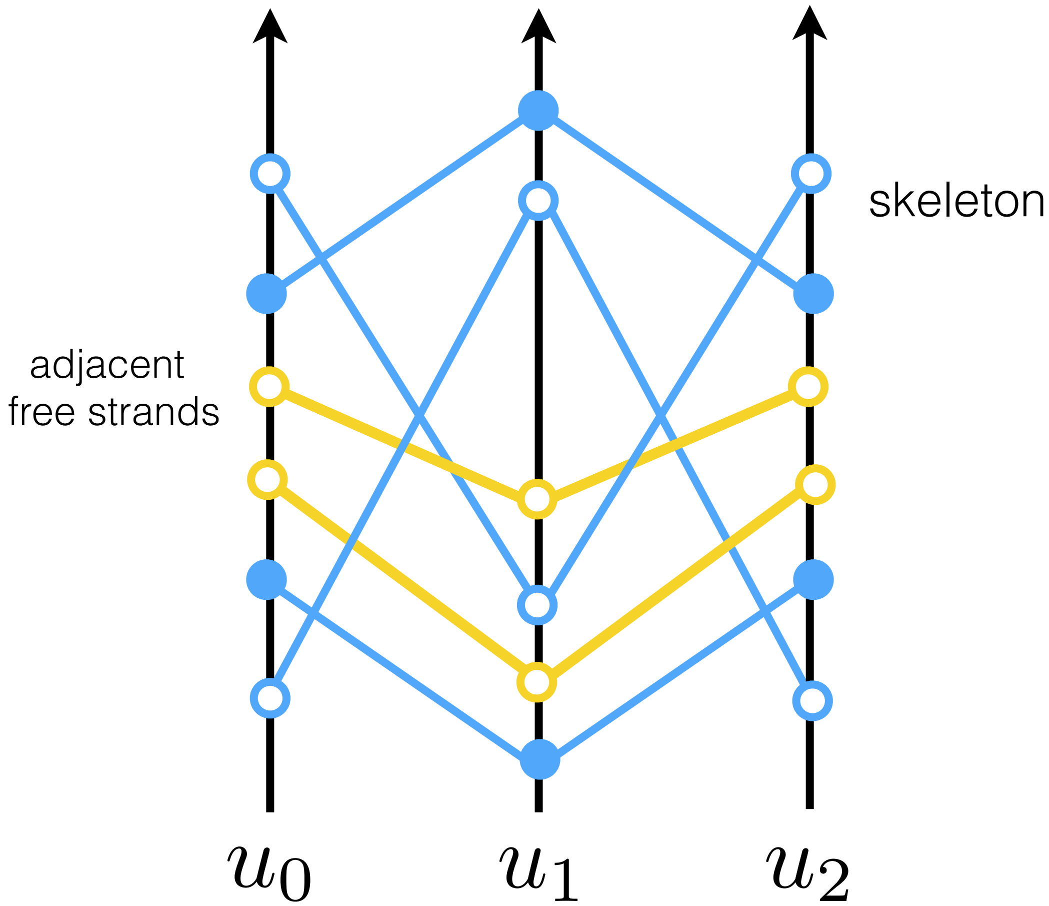

Conley index for braids gives existence theorems for PDE

# compute a Conley index



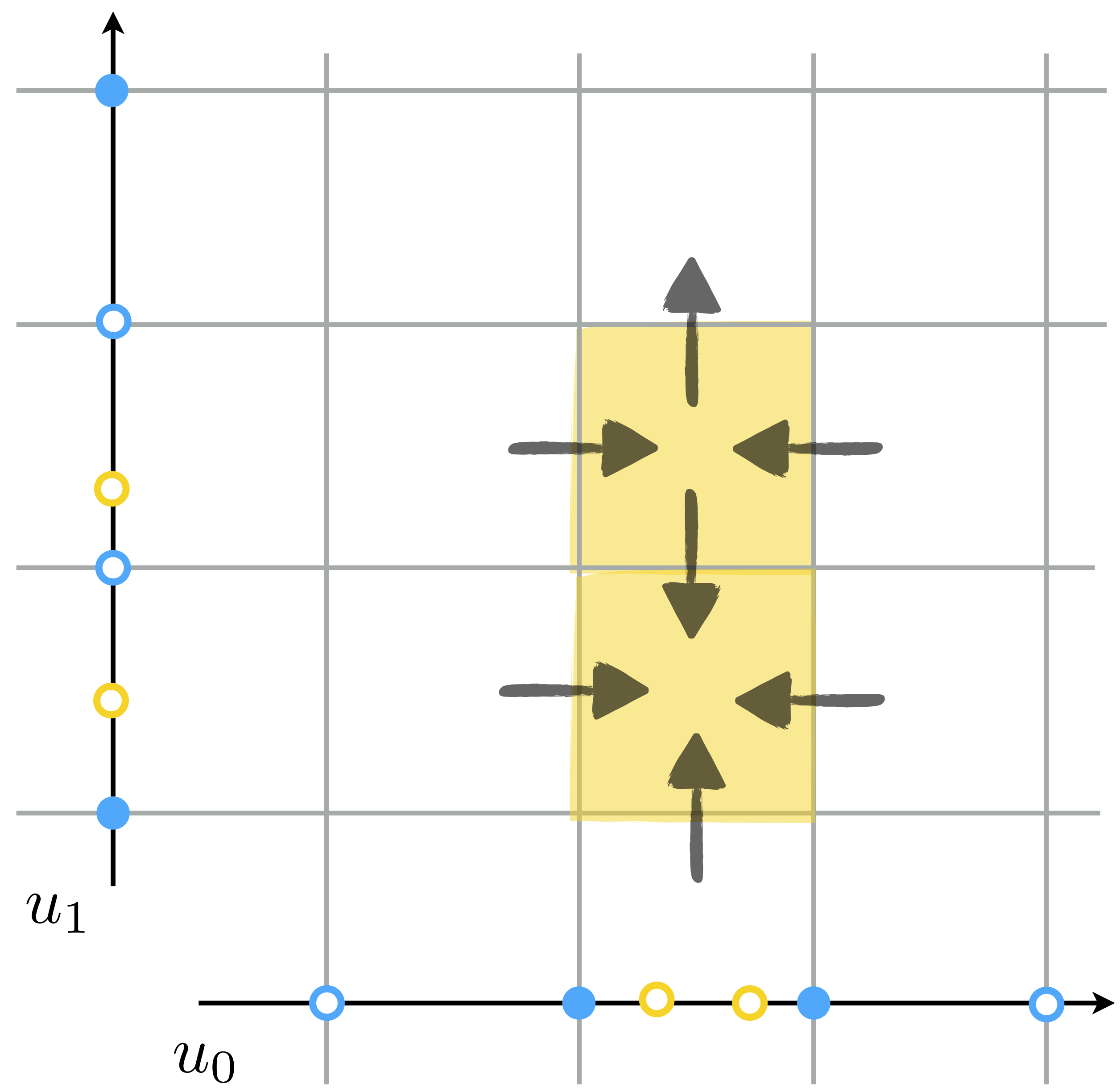

$$\partial = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$H_\bullet($  $) \cong H_\bullet($  $) \cong 0_0 \oplus \mathbb{F}_1 \oplus 0_2 \oplus \dots$




attractor-repeller pair

global dynamics

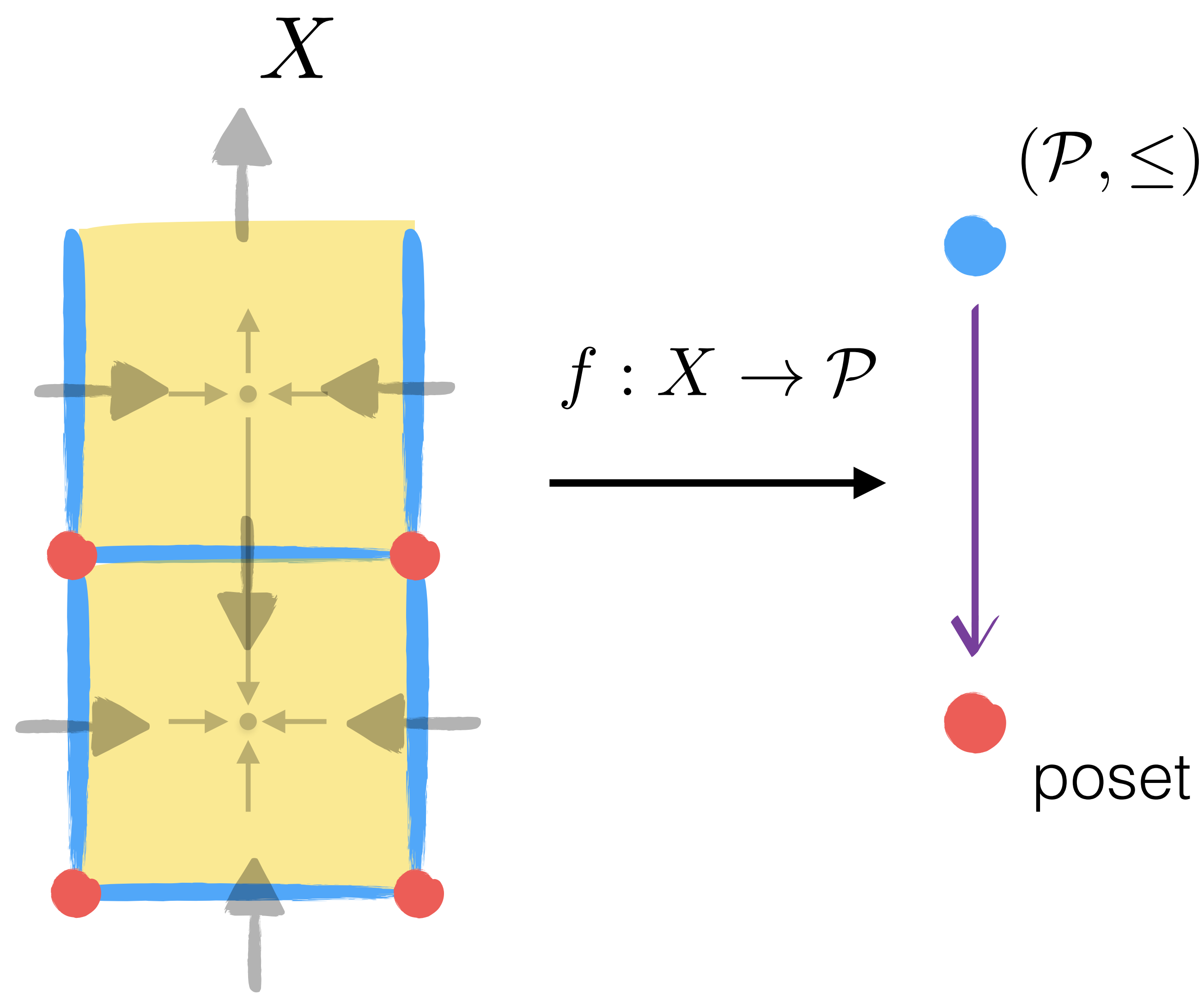


# Morse decomposition

Conley indices

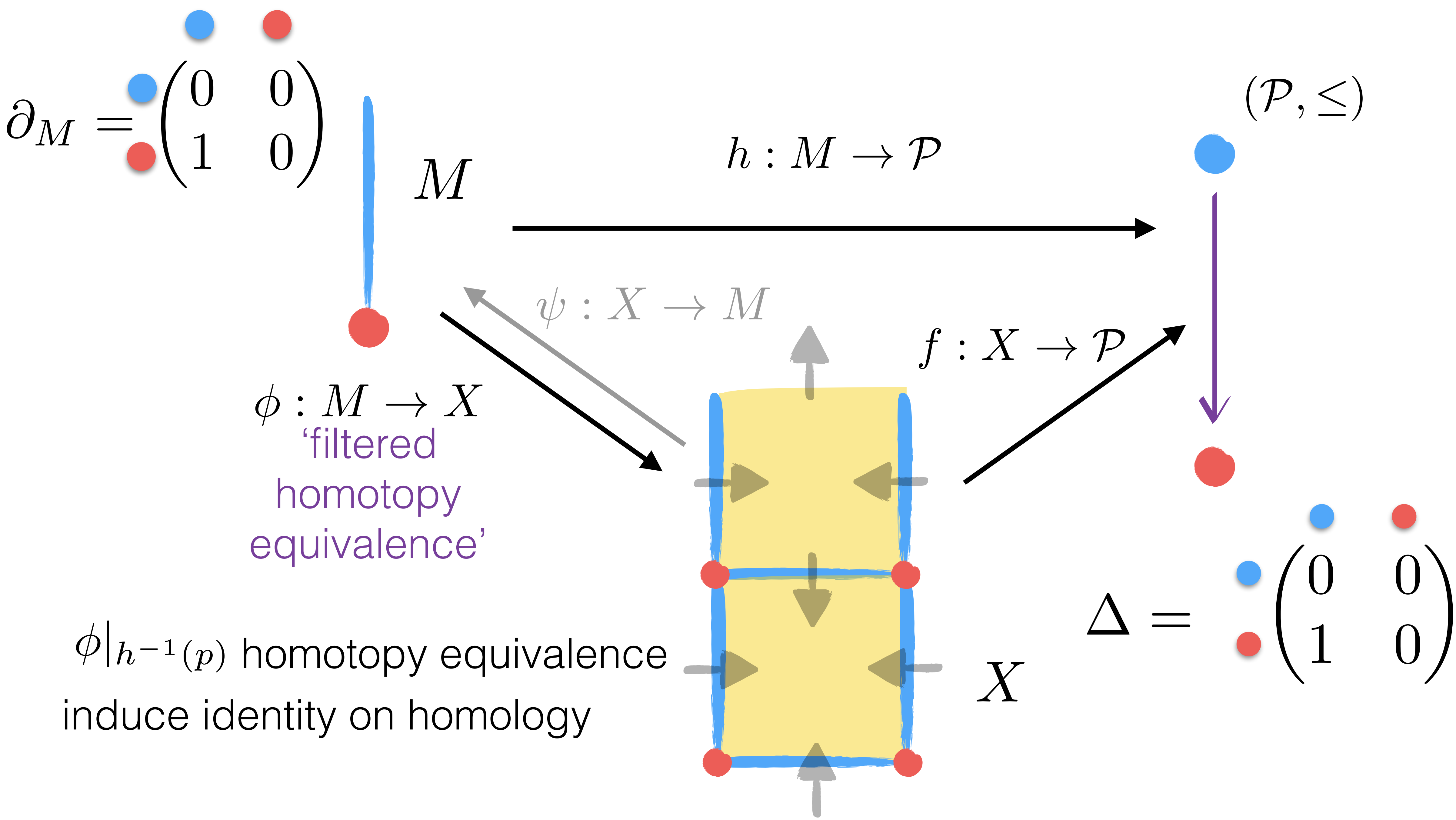
$H_\bullet(\ )$

$H_\bullet(\ )$




$$\Delta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

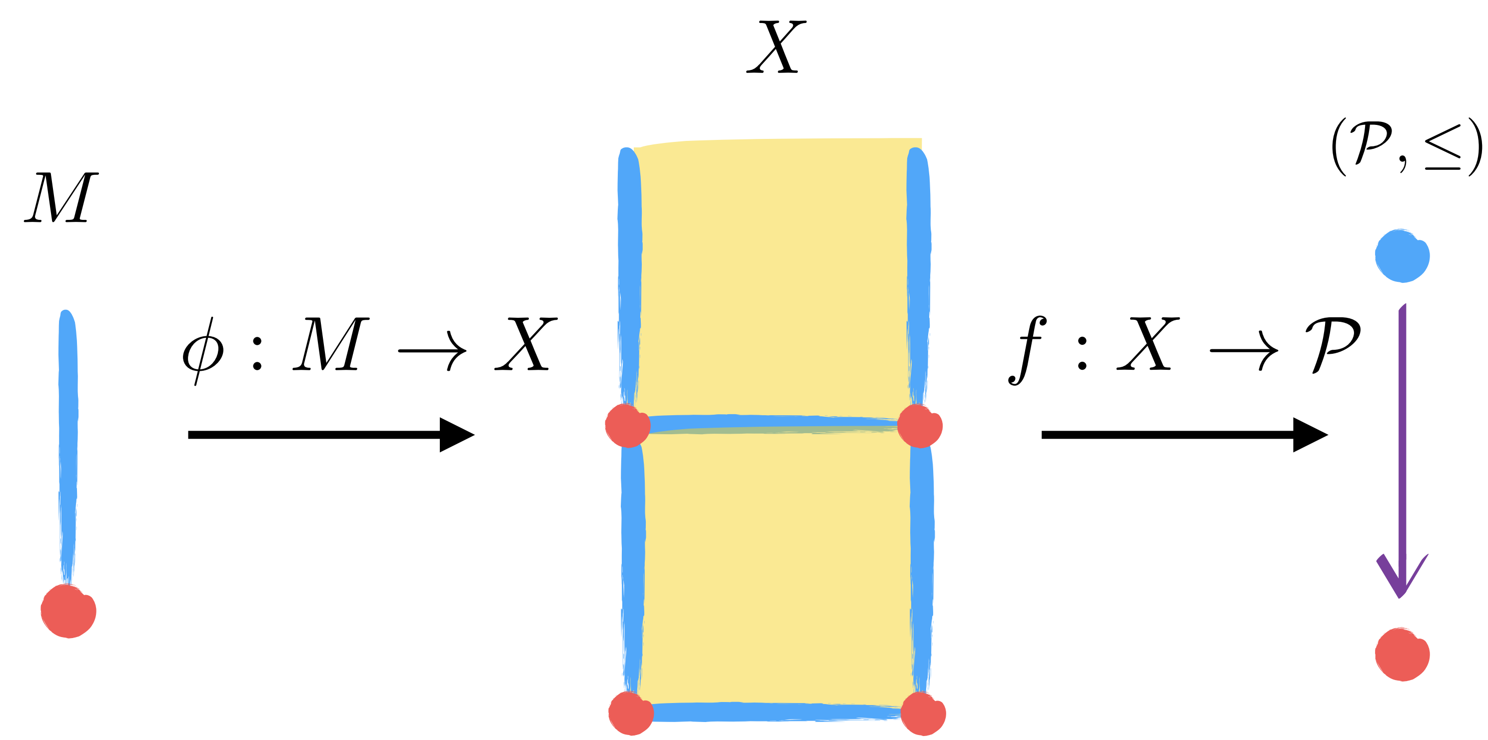
computing the connection matrix




theorem: $\partial_M|_{h^{-1}(p)} \equiv 0$ then $\partial_M$ is a connection matrix

idea: replace fibers with cells of correct homology, preserve global structure

# algorithm

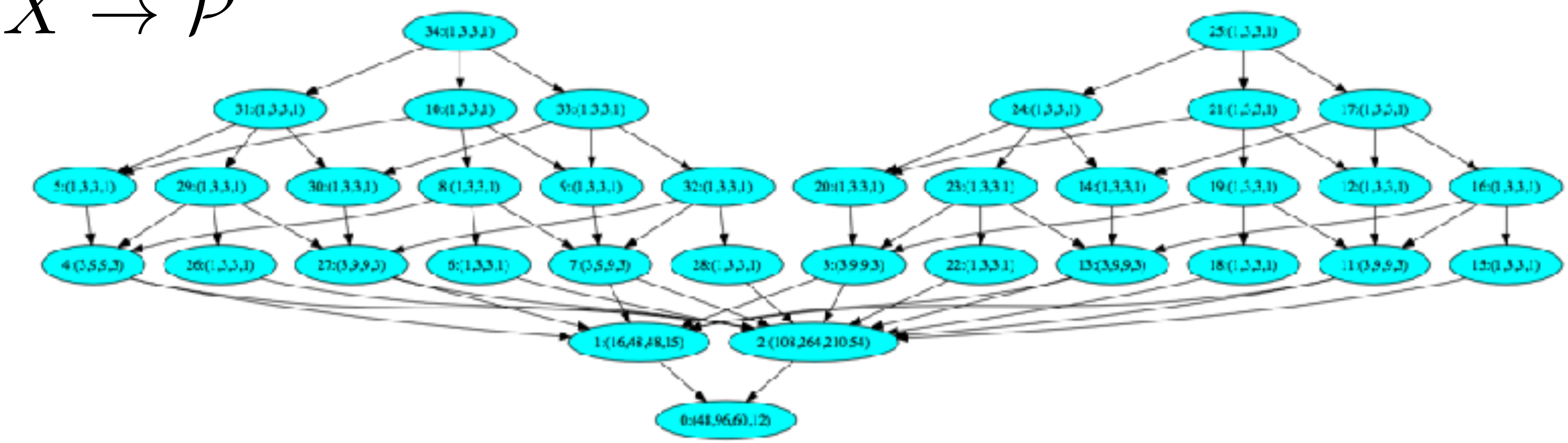



- given $f : X \to \mathcal{P}$
- use discrete Morse theory to simplify the fibers
- output $\phi, h := f \circ \phi : M \to \mathcal{P}$
  where $\partial_M$ is trivial on fibers

$$\partial_M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

python implementation (demo)

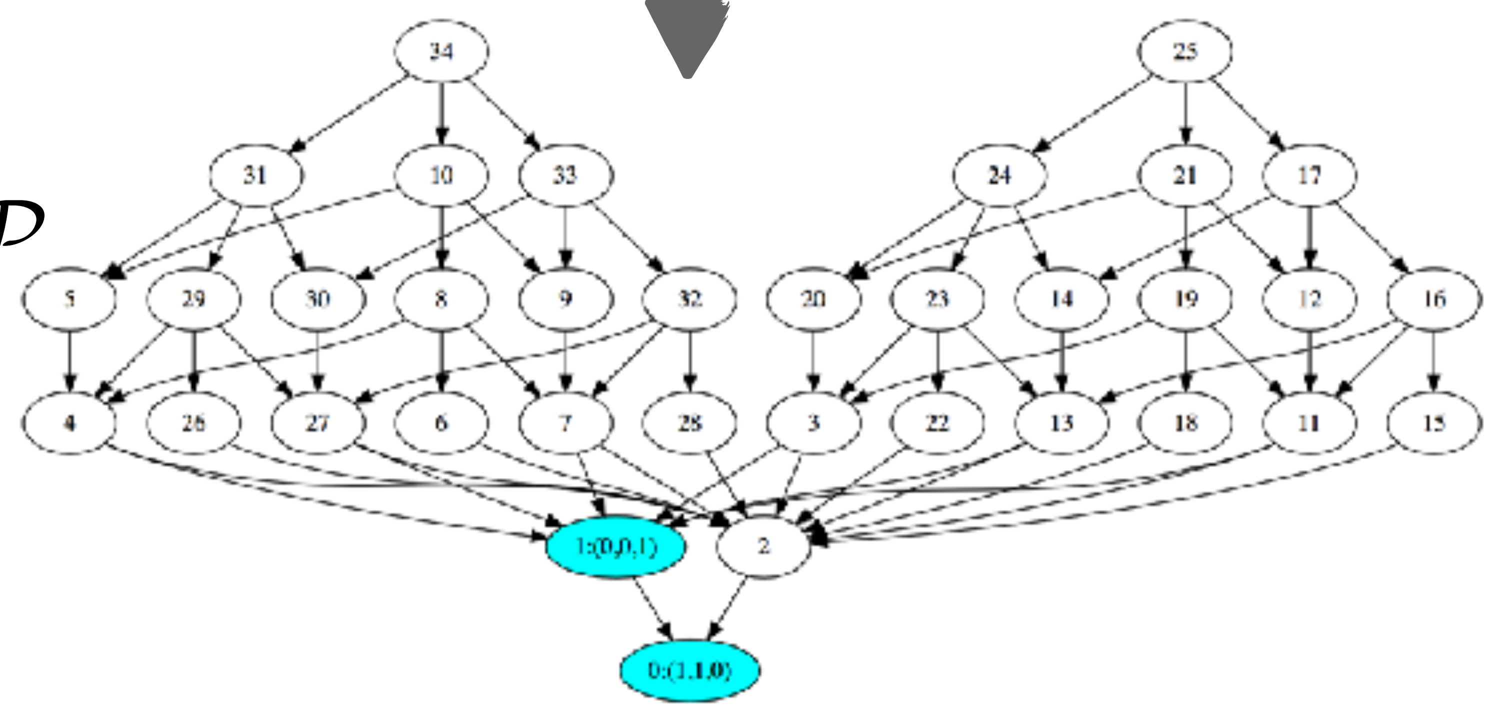
$f : X \to \mathcal{P}$



under the hood…

…discrete morse theory

$f \circ \psi : M \to \mathcal{P}$



stay tuned…

- C++ implementation chomp.rutgers.edu
- DSGRN, maps (Conley-Morse database)
- transition matrices (restriction maps of the Conley sheaf)

thank you for your attention

Conley Theory:
S. Harker
K. Mischaikow

Braids:
R. van der Vorst
M. Kramar

special thanks to MBI for hosting