Conley theory and the global dynamics of games

…from multi-agent dynamics to combinatorics and back again

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Dynamical Systems for Machine Learning

…is there a map of the talk?

dynamical systems (for ml)

- periodic orbits
- chaos theory
- bifurcation theory

Conley theory

- decomposition
- reconstruction

interlude

- combinatorial dynamics

combinatorialization

- continuous to combinatorial and back

application

- game theory/multi-agent systems

References

1. Z. Arai, W. Kalies, H. Kokubu, K. Mischaikow, H. Oka, and P. Pilarczyk. A database schema for the analysis of global dynamics of multiparameter systems. *SIAM Journal on Applied Dynamical Systems*, 8(3):757–789, 2009.
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4. J. Franks and D. Richeson. Shift equivalence and the conley index. *Transactions of the American Mathematical Society*, 352(7):3305–3322, 2000.
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6. K. Mischaikow. Conley index theory. In *Dynamical systems*, pages 119–207. Springer, 1995.
7. K. Mischaikow. Topological techniques for efficient rigorous computation in dynamics. *Acta Numerica*, 11:435–477, 2002.
8. K. Mischaikow and M. Mrozek. Chaos in the lorenz equations: a computer-assisted proof. *Bulletin of the American Mathematical Society*, 32(1):66–72, 1995.

⋮ ⋮ ⋮

dynamical systems

on butterflies in your simulation,
can proverbial straws be more catastrophic than plastic ones?
why it’s impossible to predict the weather.

given a metric space X ‘phase space’

(for the purposes of this talk) a dynamical system is a continuous map

$$f : X \to X$$

dynamical systems often include parameters:

$$f : X \times \Lambda \to X$$

typical questions one may ask of a dynamical system include:

for a particular initial condition, what happens in the long run?
how does the long run behavior depend upon the initial conditions?
…on the parameters within the model?
are there fixed points? i.e. solutions to $f(x) = x$
are there periodic orbits? i.e. solutions to $f^n(x) = x$
are there multiple attractors?

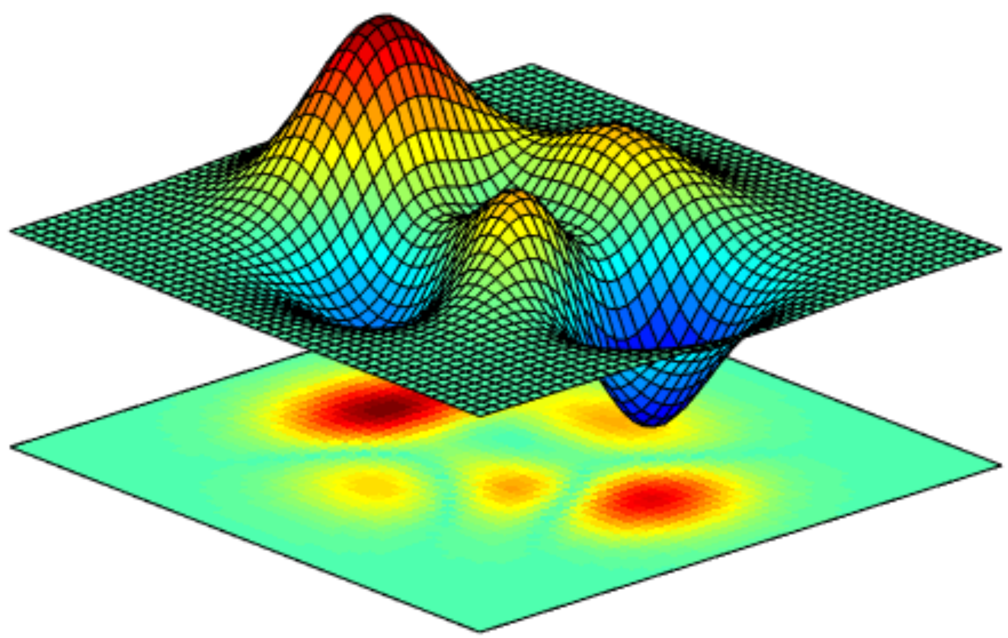
$\vdots$

dynamics can be simple (gradient-like)

$$f : X \to \mathbb{R}$$

$$\dot{x} = -\nabla f(x)$$

$$x_{n+1} = x_n - \gamma \nabla f(x_n)$$



dynamics: equilibria + heteroclinic orbits

Lee, J. D., et al. (2019). First-order methods almost always avoid strict saddle points. *Mathematical programming.*

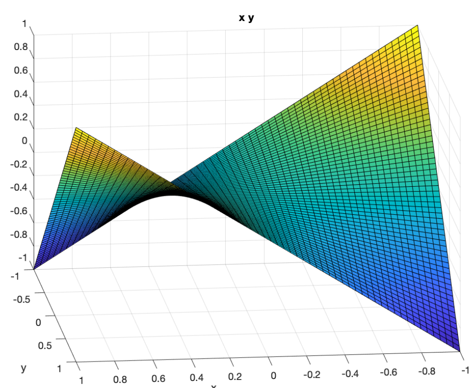
dynamics can be complex…

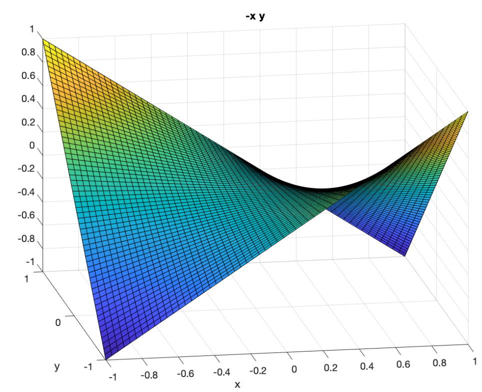
periodic orbits

dynamics can be complex

two loss functions (zero sum game)

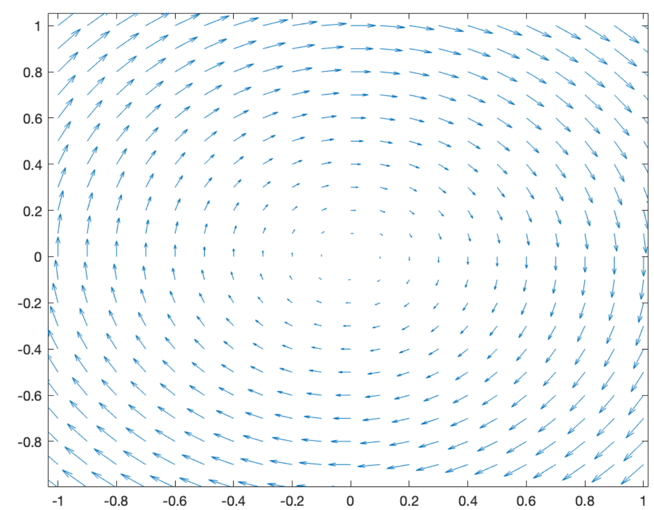
$$l_1(x, y) = xy \qquad l_2(x, y) = -xy$$





vector field:

$$\left(\frac{\partial l_1}{\partial x}, \frac{\partial l_2}{\partial y}\right) = (y, -x)$$



‘gradient descent’ on each loss

$$x_{n+1} = x_n - \gamma y_n$$
$$y_{n+1} = y_n - \gamma x_n$$

instantiation: Generative Adversarial Networks (GANs)

Goodfellow, I. et al. (2014). Generative Adversarial Networks. NIPS 2014.

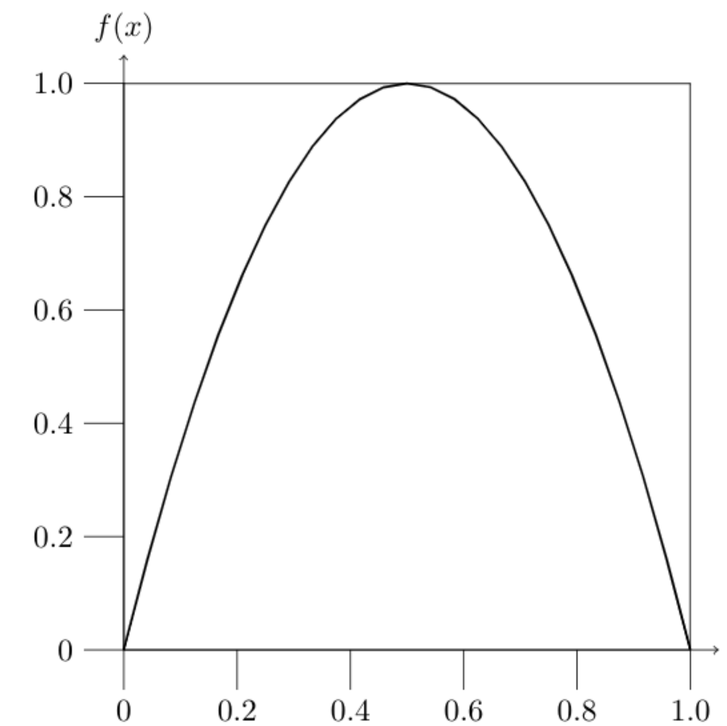
dynamics can be complex II

$$f\colon [0,1] \to [0,1]$$

Logistic map

$$f(x) = 4x(1-x)$$

$$x_{n+1} = f(x_n)$$



dynamics can be complex II

$$f : [0,1] \to [0,1] \qquad f(x) = 4x(1-x)$$

Logistic map

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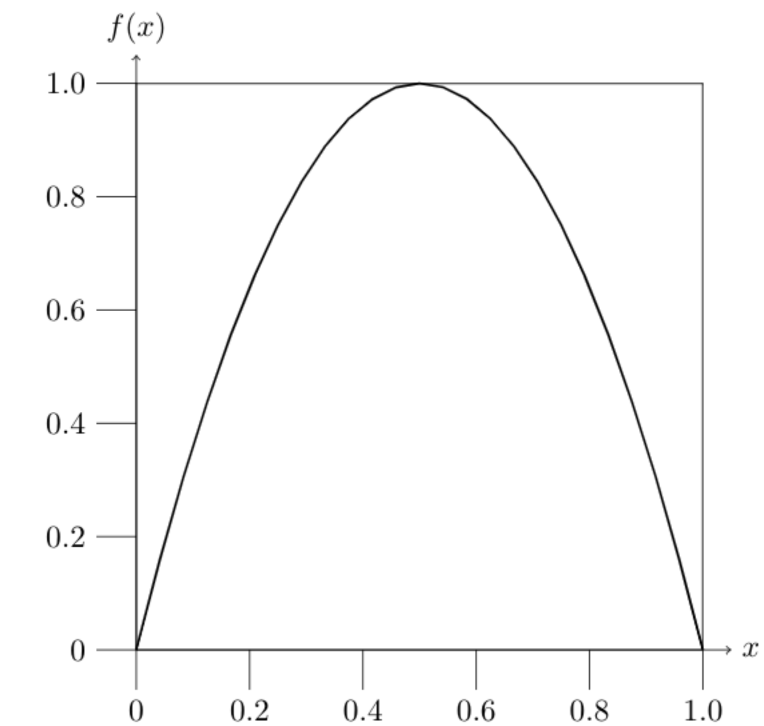


a flap of a butterfly's wings in Brazil…

deviations in
the 15th decimal

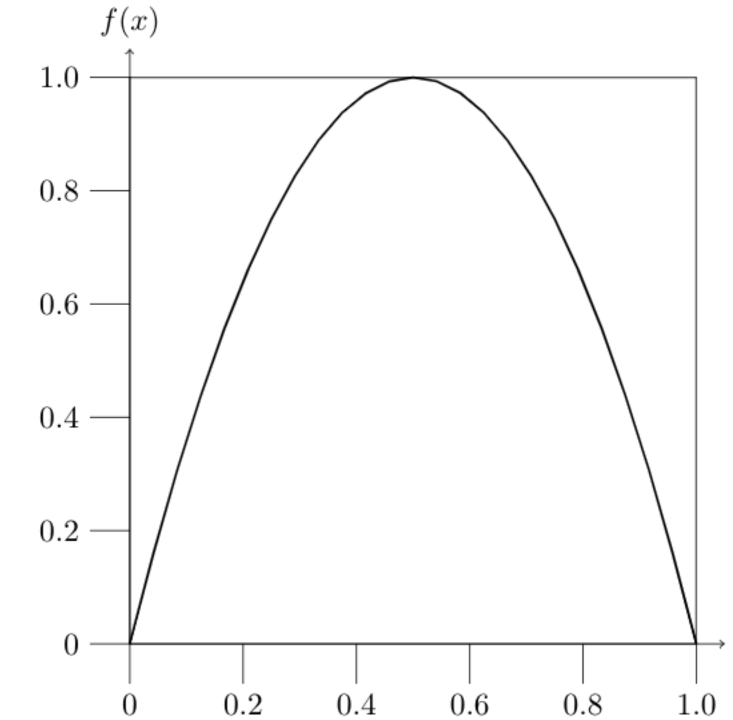
$x_0 = 0.333333333333333$ $\qquad$ $x_0 = 0.333333333333330$

dynamics can be complex II

$$f\colon [0,1] \to [0,1] \qquad f(x) = 4x(1-x)$$

Logistic map

$$x_{n+1} = f(x_n)$$



a flap of a butterfly's wings in Brazil…

deviations in the 15th decimal

$x_0 = 0.33333333333333\boxed{3}$ $\quad$ $x_0 = 0.33333333333333\boxed{0}$

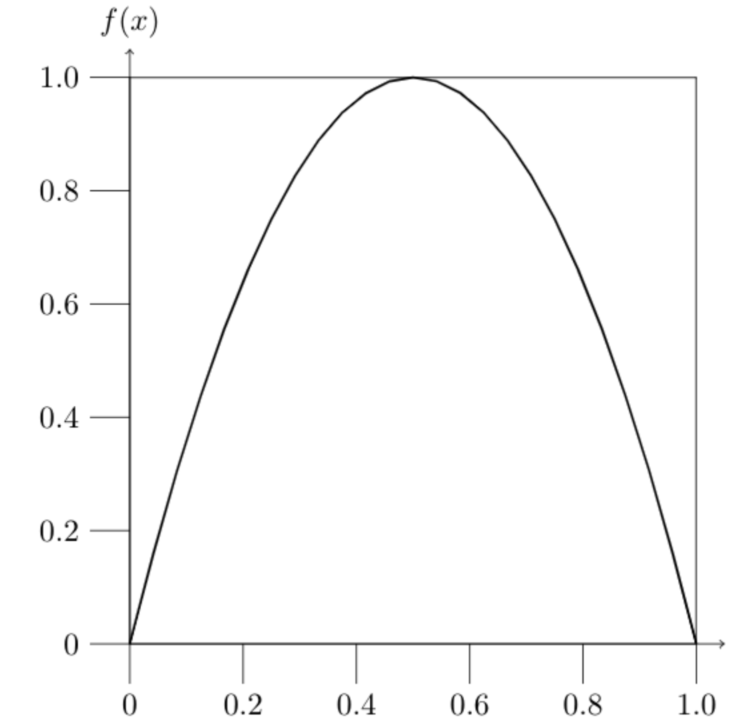
$x_1 = 0.88888888888888\boxed{9}$ $\quad$ $x_1 = 0.88888888888888\boxed{5}$

dynamics can be complex II

$$f : [0,1] \to [0,1] \qquad f(x) = 4x(1-x)$$

Logistic map

$$x_{n+1} = f(x_n)$$



a flap of a butterfly's wings in Brazil…

deviations in
the 15th decimal

$x_0 = 0.3333333333333333$ $\qquad$ $x_0 = 0.3333333333333330$

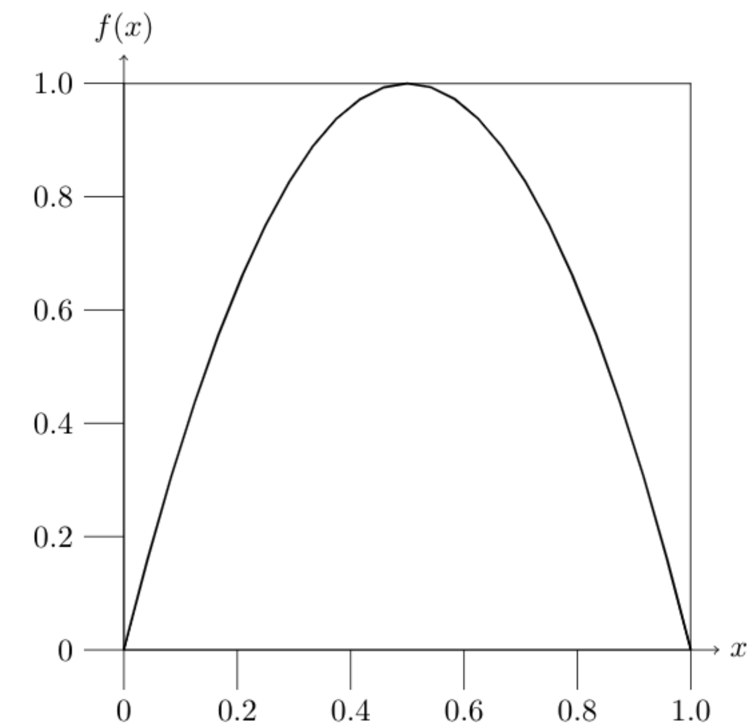
$x_1 = 0.8888888888888889$ $\qquad$ $x_1 = 0.8888888888888885$

$x_2 = 0.3950617283950662$ $\qquad$ $x_2 = 0.3950617283950675$

dynamics can be complex II

$$f: [0,1] \to [0,1] \qquad f(x) = 4x(1-x)$$

Logistic map $\qquad x_{n+1} = f(x_n)$



a flap of a butterfly's wings in Brazil…

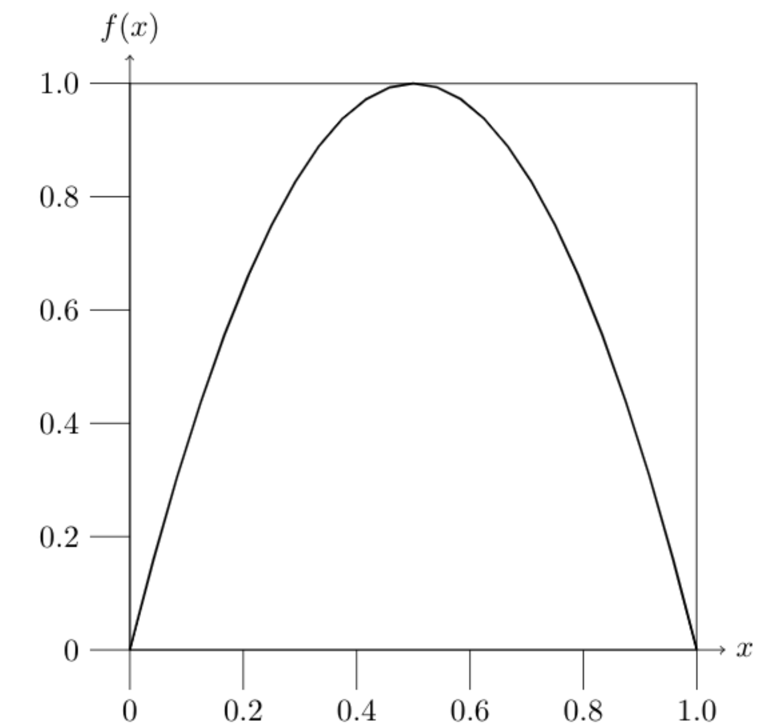
deviations in the 15th decimal

$$x_0 = 0.3333333333333333 \qquad x_0 = 0.3333333333333330$$
$$x_1 = 0.8888888888888889 \qquad x_1 = 0.8888888888888885$$
$$x_2 = 0.395061728395062 \qquad x_2 = 0.395061728395075$$
$$x_3 = 0.9559518366610273 \qquad x_3 = 0.9559518366610284$$

dynamics can be complex II

$$f\colon [0,1] \to [0,1] \qquad f(x) = 4x(1-x)$$

Logistic map $\qquad x_{n+1} = f(x_n)$



a flap of a butterfly's wings in Brazil…

deviations in
the 15th decimal

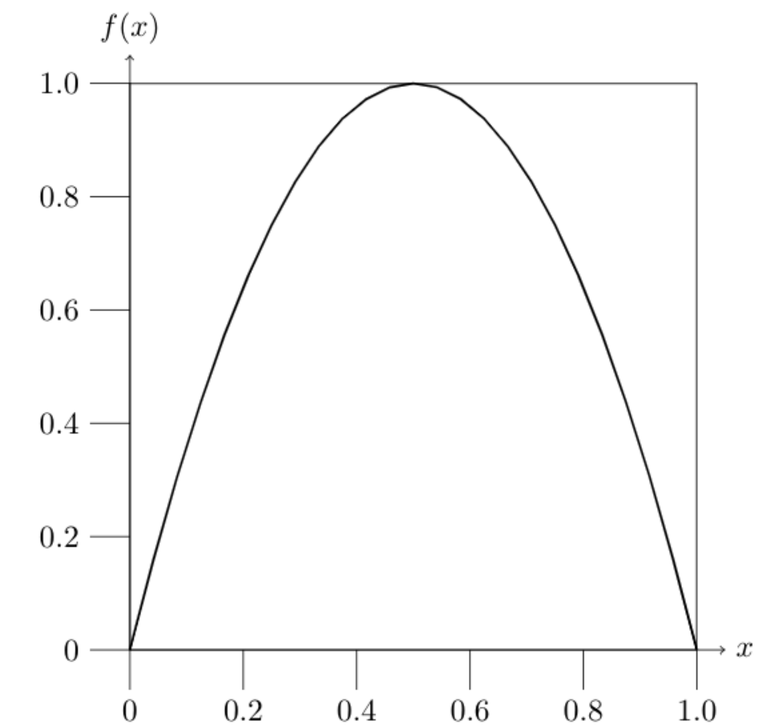
$$\begin{aligned} x_0 &= 0.333333333333333\boxed{3} \\ x_1 &= 0.888888888888888\boxed{9} \\ x_2 &= 0.3950617283950\boxed{62} \\ x_3 &= 0.9559518366102\boxed{73} \\ &\vdots \\ x_{15} &= 0.695026128\boxed{241317} \end{aligned} \qquad \begin{aligned} x_0 &= 0.333333333333333\boxed{0} \\ x_1 &= 0.888888888888888\boxed{5} \\ x_2 &= 0.3950617283950\boxed{75} \\ x_3 &= 0.9559518366102\boxed{84} \\ &\vdots \\ x_{15} &= 0.695026128\boxed{347429} \end{aligned}$$

dynamics can be complex II

$$f\colon [0,1] \to [0,1] \qquad f(x) = 4x(1-x)$$

Logistic map

$$x_{n+1} = f(x_n)$$



a flap of a butterfly's wings in Brazil…

deviations in the 15th decimal

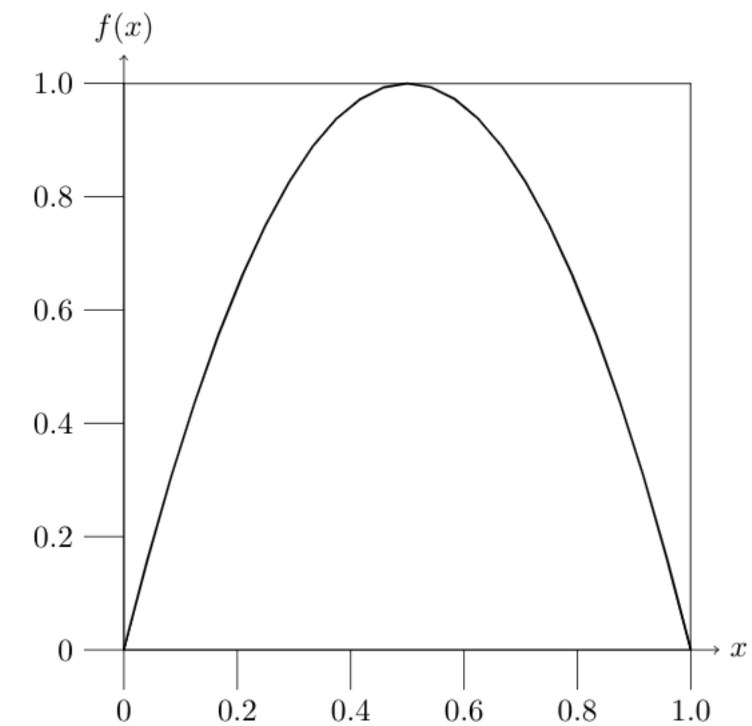
$$\begin{aligned} x_0 &= 0.333333333333333 \\ x_1 &= 0.888888888888889 \\ x_2 &= 0.395061728395062 \\ x_3 &= 0.955951836661027 \\ &\vdots \\ x_{15} &= 0.695026128241317 \\ &\vdots \\ x_{49} &= 0.071160322456580 \end{aligned}$$

$$\begin{aligned} x_0 &= 0.333333333333330 \\ x_1 &= 0.888888888888885 \\ x_2 &= 0.395061728395075 \\ x_3 &= 0.955951836661028 \\ &\vdots \\ x_{15} &= 0.695026128347429 \\ &\vdots \\ x_{49} &= 0.906436654059206 \end{aligned}$$

dynamics can be complex II

$$f : [0,1] \to [0,1] \qquad f(x) = 4x(1-x)$$

Logistic map $\qquad x_{n+1} = f(x_n)$



a flap of a butterfly's wings in Brazil…

deviations in
the 15th decimal

$$\begin{aligned} x_0 &= 0.333333333333333 \\ x_1 &= 0.888888888888889 \\ x_2 &= 0.395061728395062 \\ x_3 &= 0.955951836610273 \\ &\vdots \\ x_{15} &= 0.695026128241317 \\ &\vdots \\ x_{49} &= 0.071160322456580 \end{aligned} \qquad \begin{aligned} x_0 &= 0.333333333333330 \\ x_1 &= 0.888888888888885 \\ x_2 &= 0.395061728395075 \\ x_3 &= 0.955951836610284 \\ &\vdots \\ x_{15} &= 0.695026128347429 \\ &\vdots \\ x_{49} &= 0.906436654059206 \end{aligned}$$

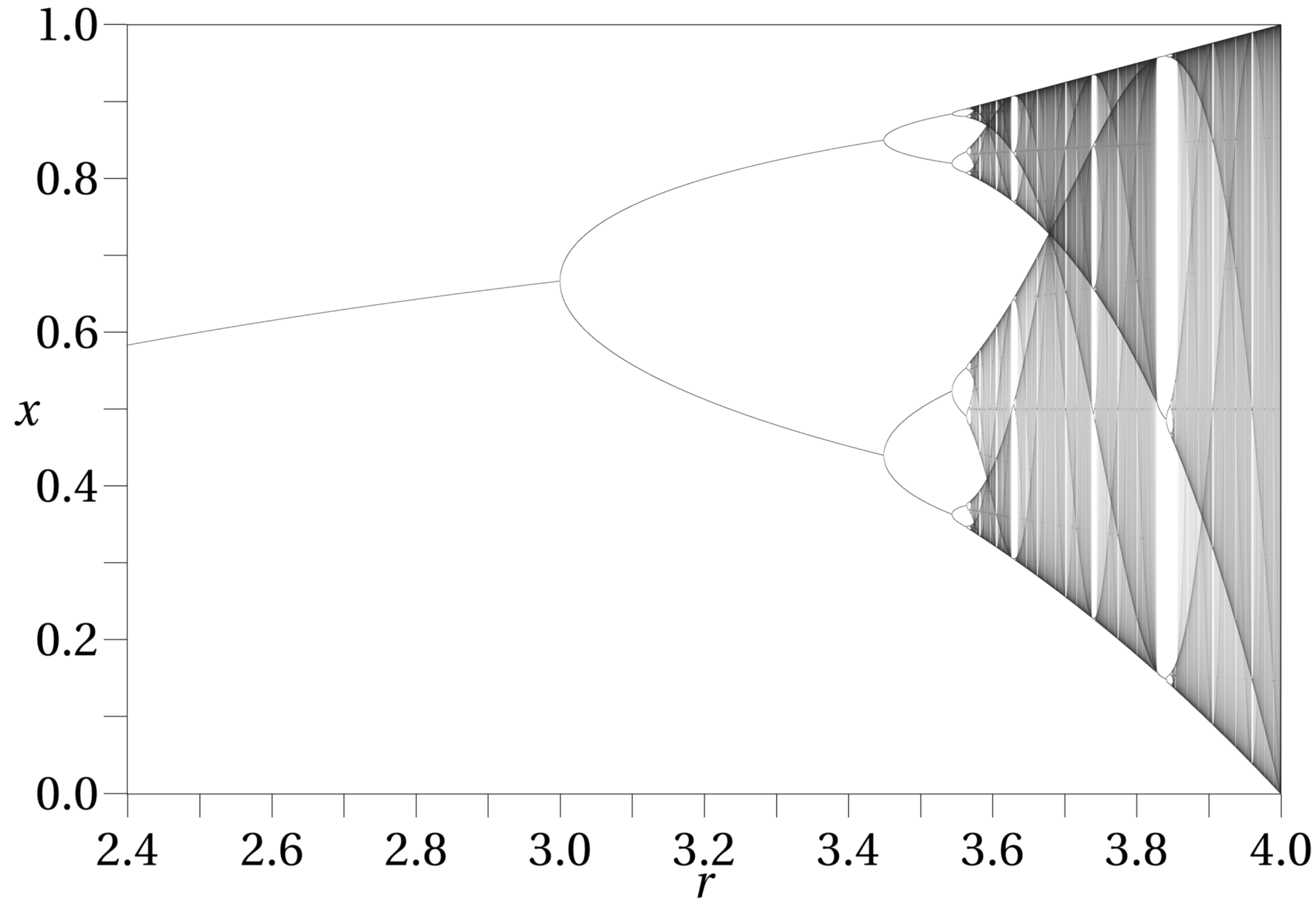
deviation is now on the
order of the entire phase space

…sets off a tornado in Texas.

dynamics can be complex III

$$x_{n+1} = f_r(x) = rx_n(1 - x_n)$$

r is a model parameter



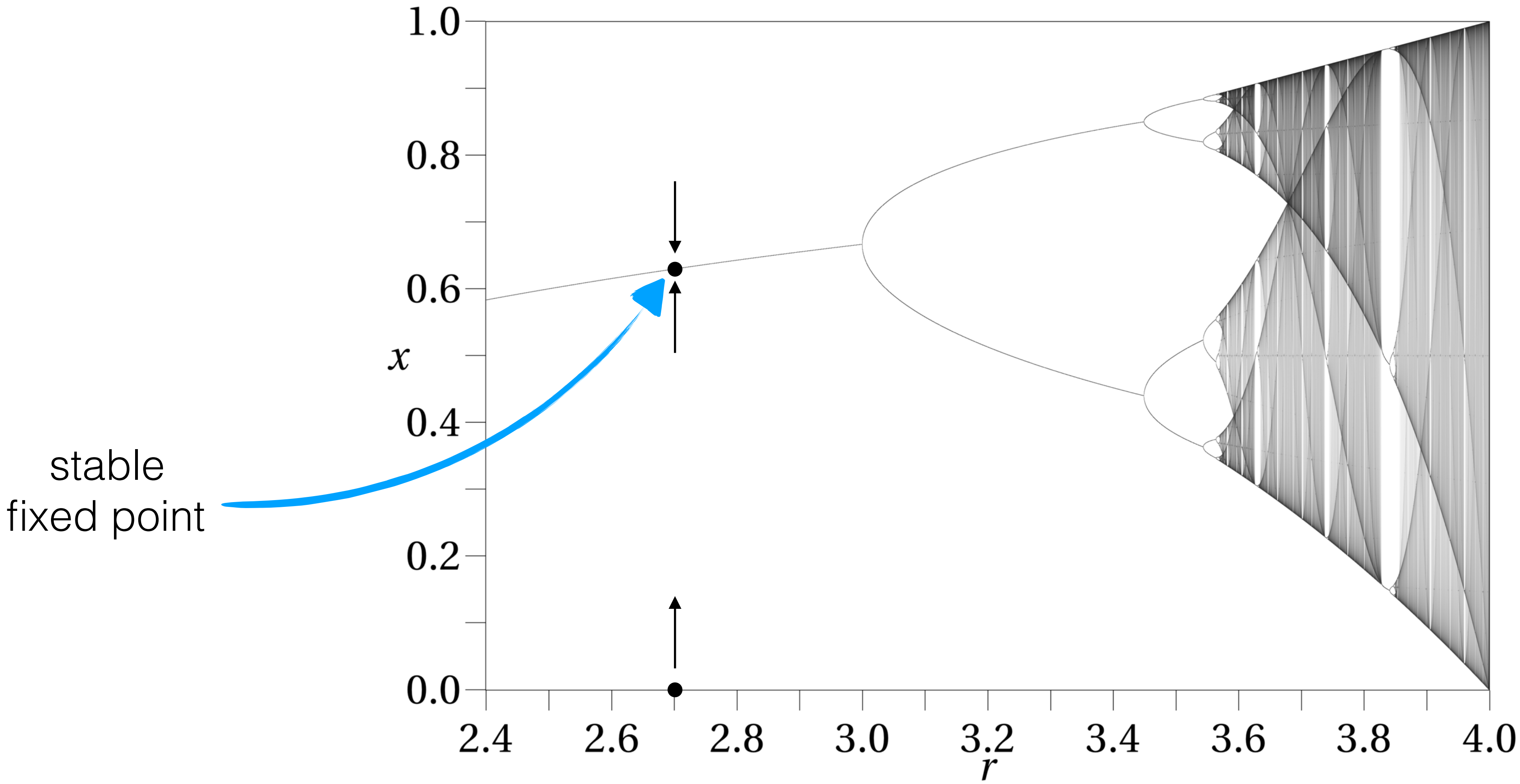
‘bifurcation diagram’

plotted are values visited asymptotically from almost all initial conditions

dynamics can be complex III

$$x_{n+1} = f_r(x) = r x_n (1 - x_n)$$

r is a model parameter



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dynamics can be complex III

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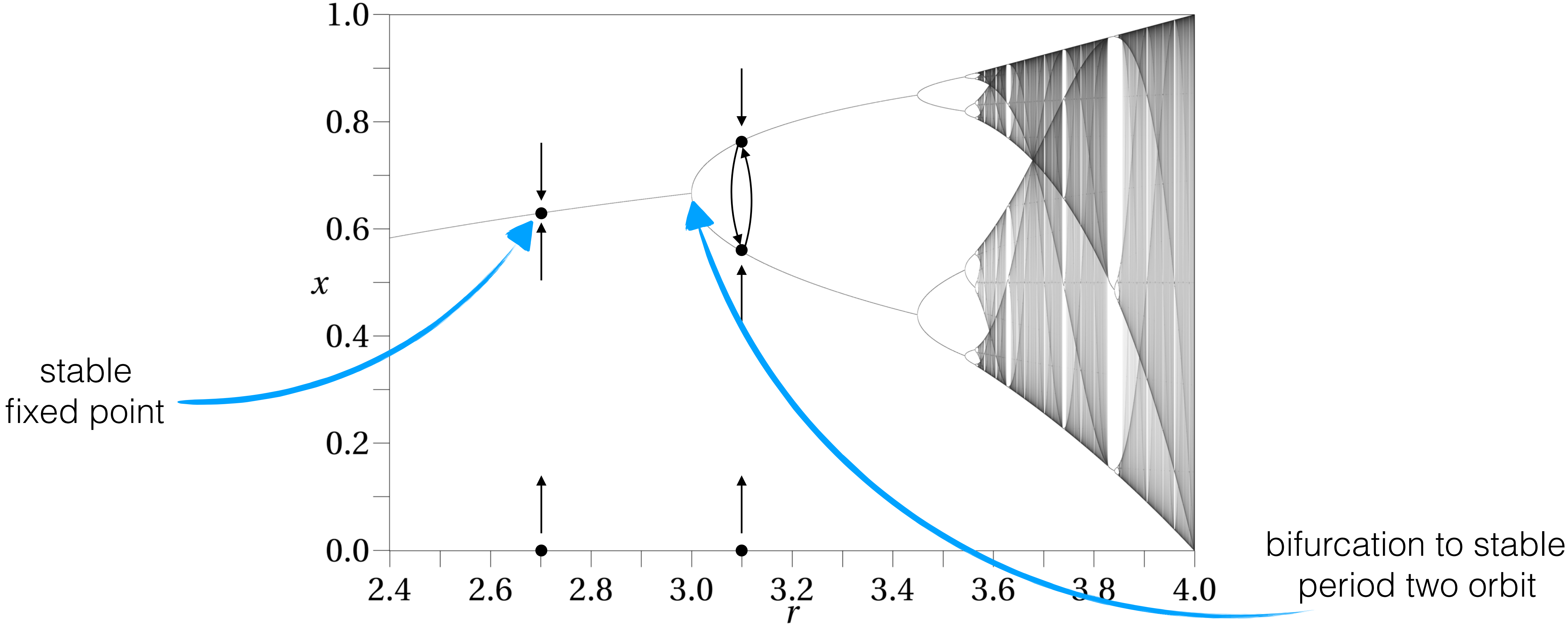

'bifurcation diagram'

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dynamics can be complex III

$$x_{n+1} = f_r(x) = rx_n(1 - x_n)$$

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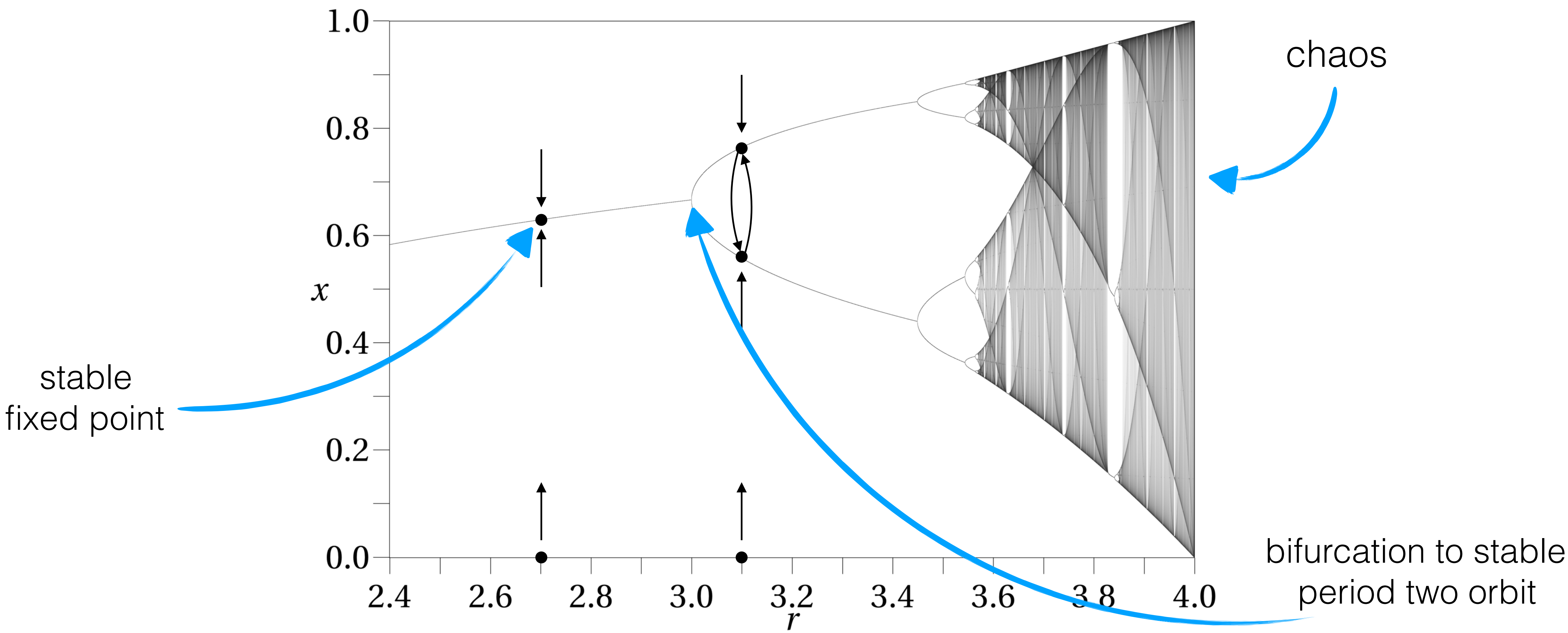


bifurcations: a single straw can break the camel's back

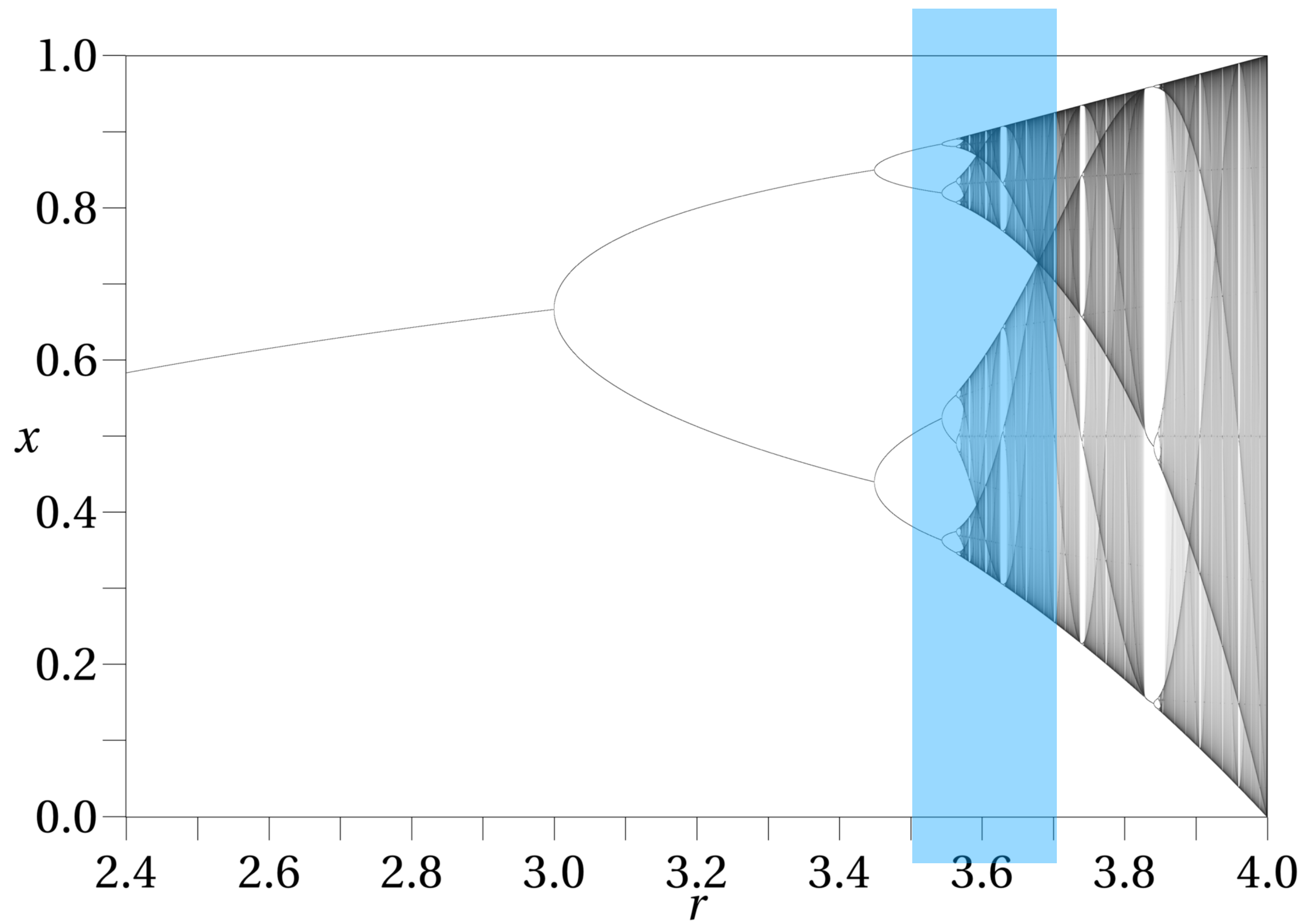
that is, a small perturbation in parameters can drastically change dynamics *a thought experiment*

suppose… that the logistic map is the perfect model (say, for the onset of turbulence), and

suppose… that one can perform a perfect numerical simulation…

…with high probability, we will still draw the wrong conclusions.

suppose an experimentalist can measure ***r*** to within one decimal place



any computation probably will suggest the wrong dynamics

Conley's theory

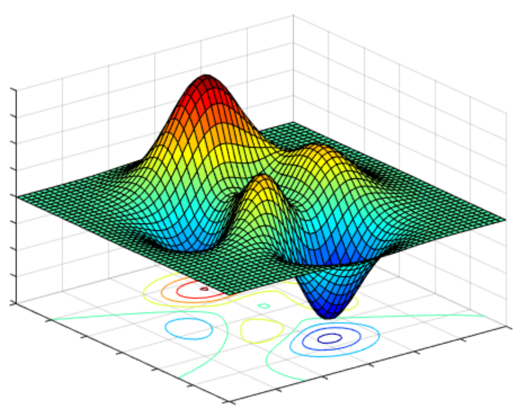
decomposition + reconstruction

'…if such rough equations are to be of use it is necessary to study them in rough terms.'
C. Conley

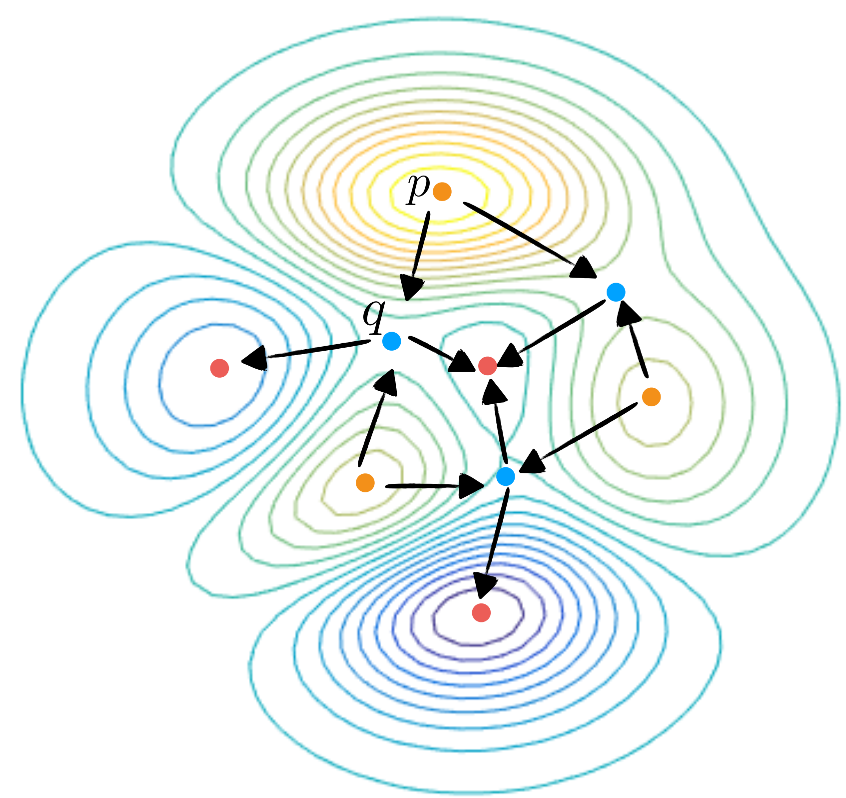
decomposition

Morse theory as a rapid prototype,
Morse graphs for your dynamical system on a napkin,
how to stop worrying and learn to be robust to perturbations.

returning to simple (Morse-theoretic) dynamics

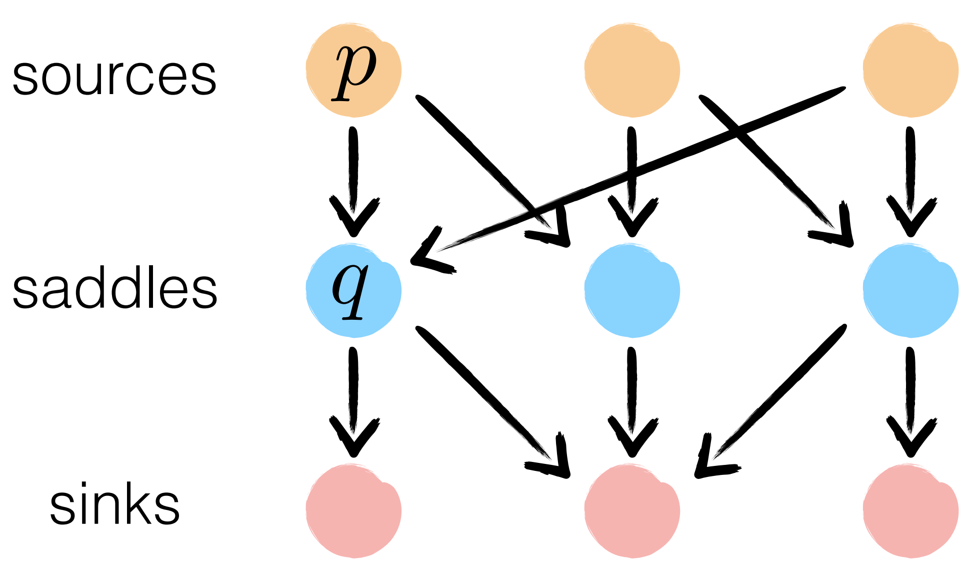


dynamics: equilibria + heteroclinic orbits



global dynamics are organized by a graph

Morse graph



fixed points become vertices

edge set satisfies the property:

if there is an orbit from ***p*** to ***q***,
then there is a path from ***p*** to ***q*** in the Morse graph

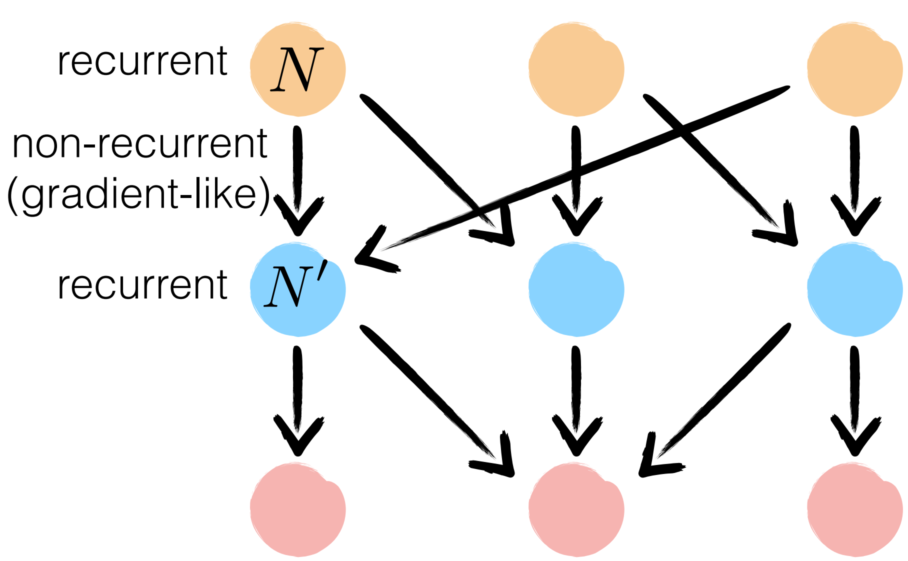
(isolating) neighborhoods instead of fixed points

a choice of neighborhoods is a choice of…
resolution

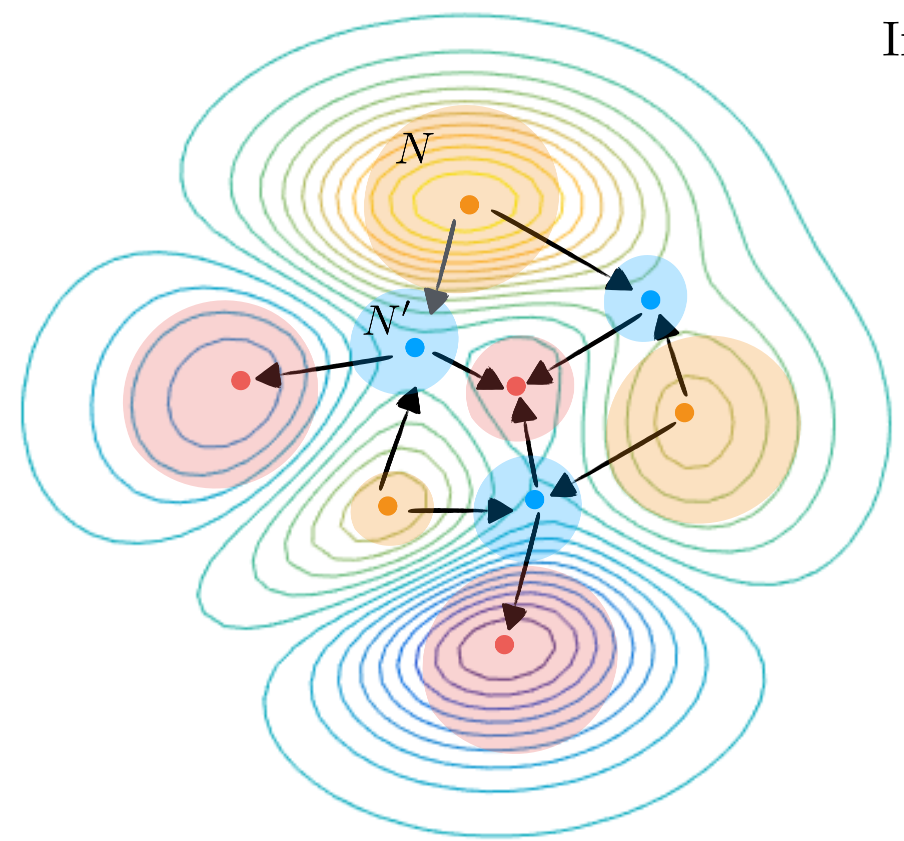
invariant dynamics must be
isolated within neighborhood

$$\mathrm{Inv}(N, f) \subset \mathrm{int}(N)$$

Morse graph



at this resolution the
decomposition remains the same

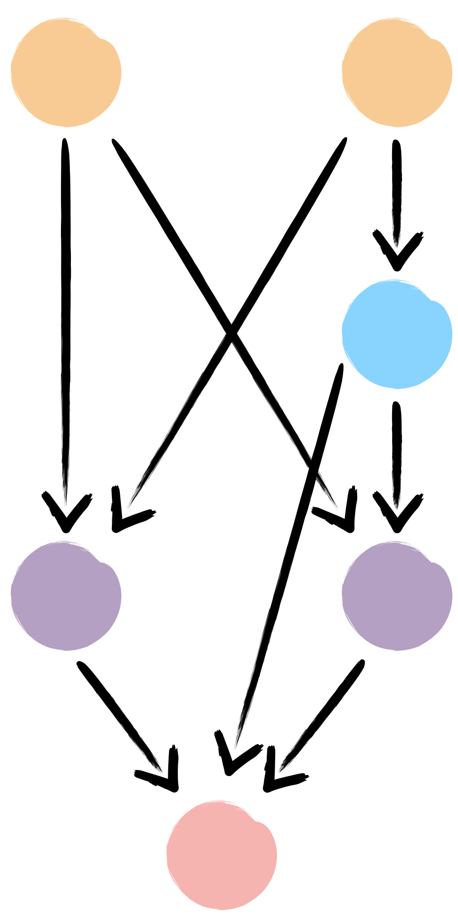


neighborhoods become vertices
edge set satisfies the property:

if there is an orbit from ***N*** to ***N`***,
then there is a path from ***N*** to ***N`*** in the Morse graph

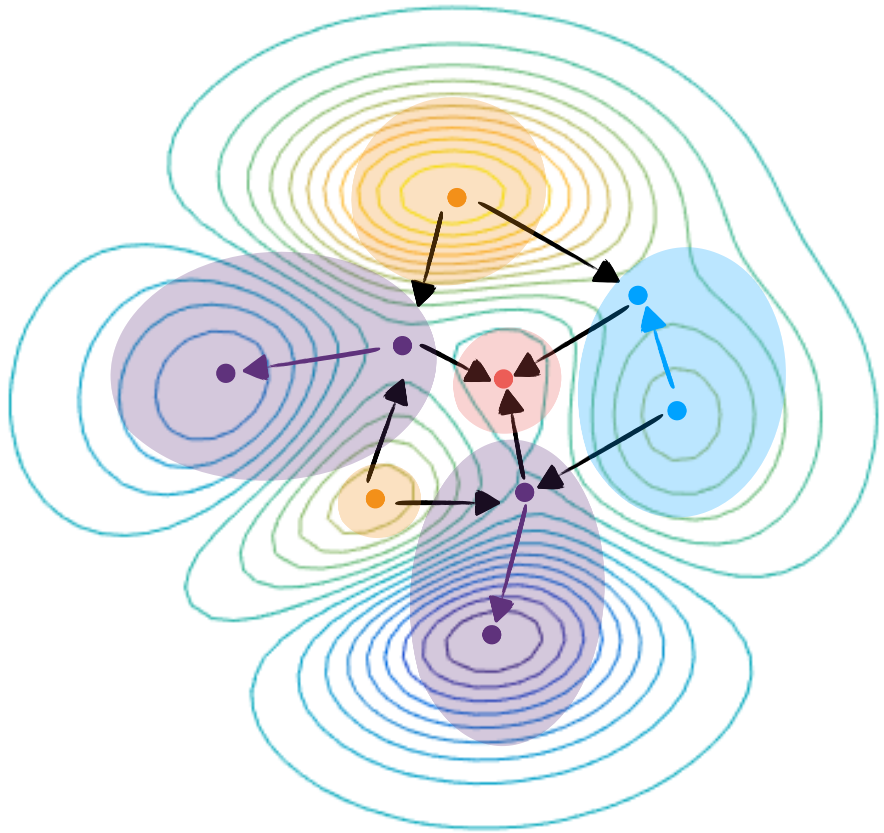
a choice of neighborhoods is a choice of… resolution

choosing different neighborhoods



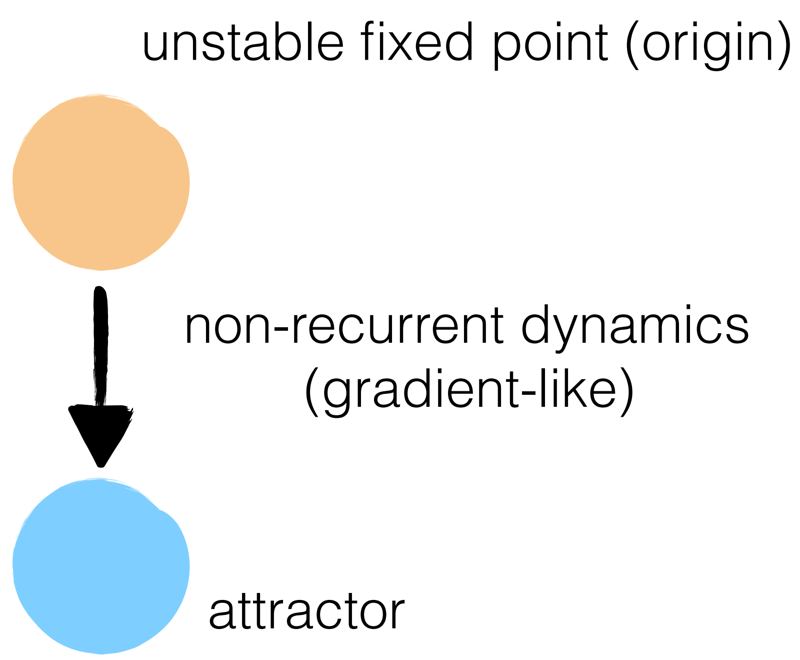
invariant dynamics must be isolated within neighborhood

$\mathrm{Inv}(N, f) \subset \mathrm{int}(N)$

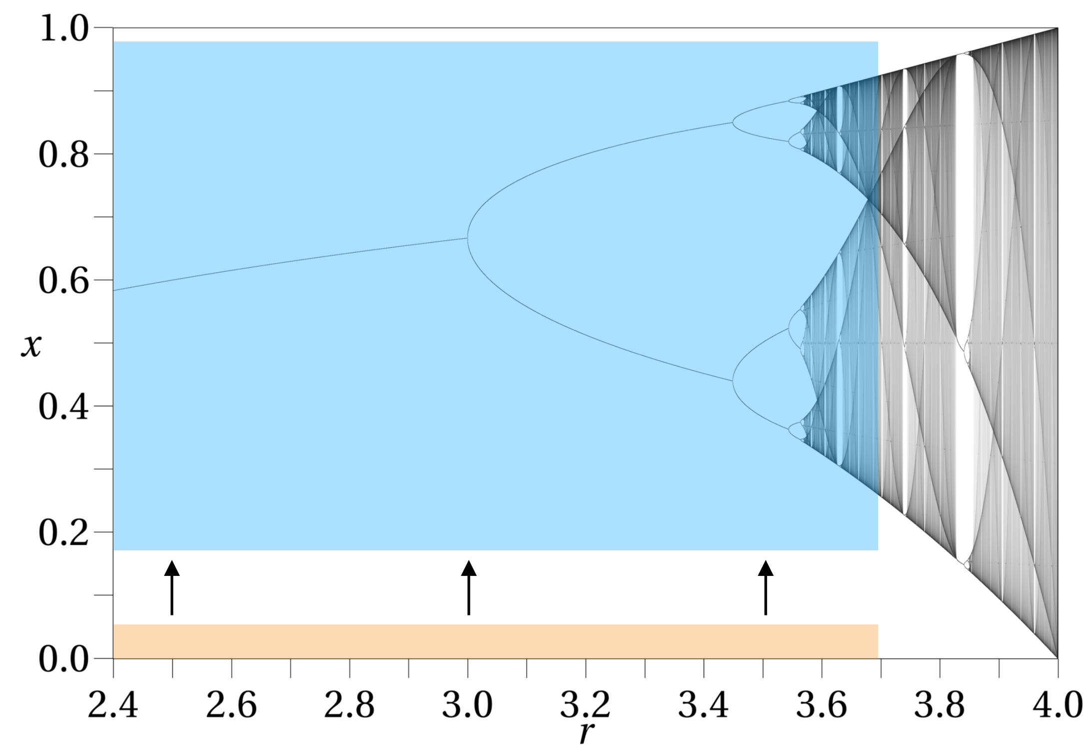


the graph is robust with respect to perturbation of the system

decompose dynamics over a set of parameters



Morse graph



a Morse graph …

- is robust (valid) over the set of parameters
- compact representation of global dynamics
- capable of capturing complex dynamics

reconstruction

invariant sets leave algebraic topological footprints,
Conley-Morse graphs = Morse graphs + Conley indices,
a one-dimensional zoo.

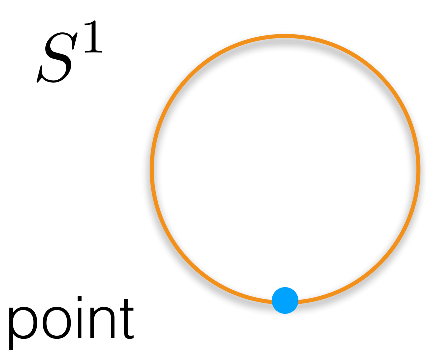
what is…homology?

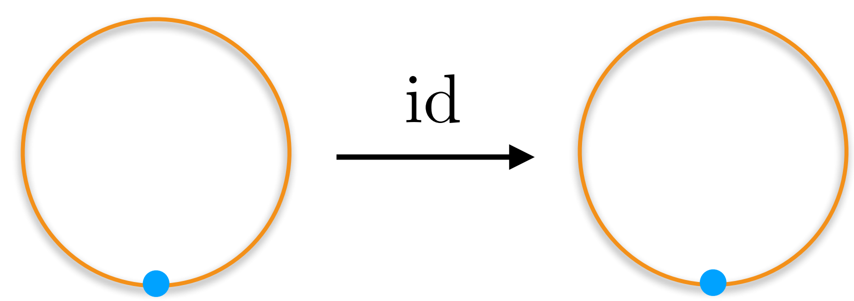
relative homology is a mapping (functor)
from pairs of topological spaces and continuous maps
to vector spaces and linear maps

topology

pair of topological spaces

continuous maps





$H_\bullet(\cdot)$

algebra

pair of vector spaces

linear maps

$$\big(H_0(S^1,\{\cdot\}), H_1(S^1,\{\cdot\})\big) = (0, \mathbb{F})$$

dimension 0 dimension 1

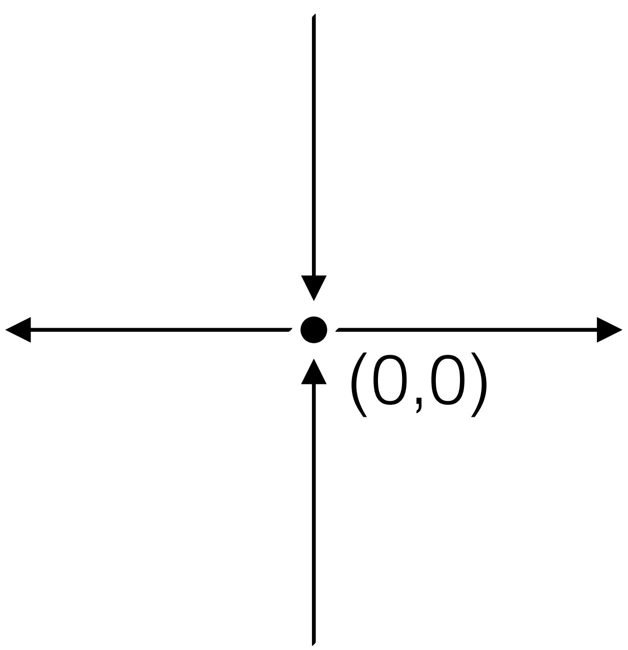
one dimensional 'hole'

$$H_\bullet(\mathrm{id}) = \big(0 \to 0, \mathbb{F} \xrightarrow{\mathrm{id}} \mathbb{F}\big)$$

what is…a Conley index?

- stable algebraic-topological invariant of invariant sets
- for continuous-time flows or discrete-time maps
- coarsely quantifies unstable dynamics

one-dimensional saddle



$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ \frac{1}{2}y \end{pmatrix}$$

Conley index is represented by the tuple

$$(0, \mathrm{id}, 0)$$

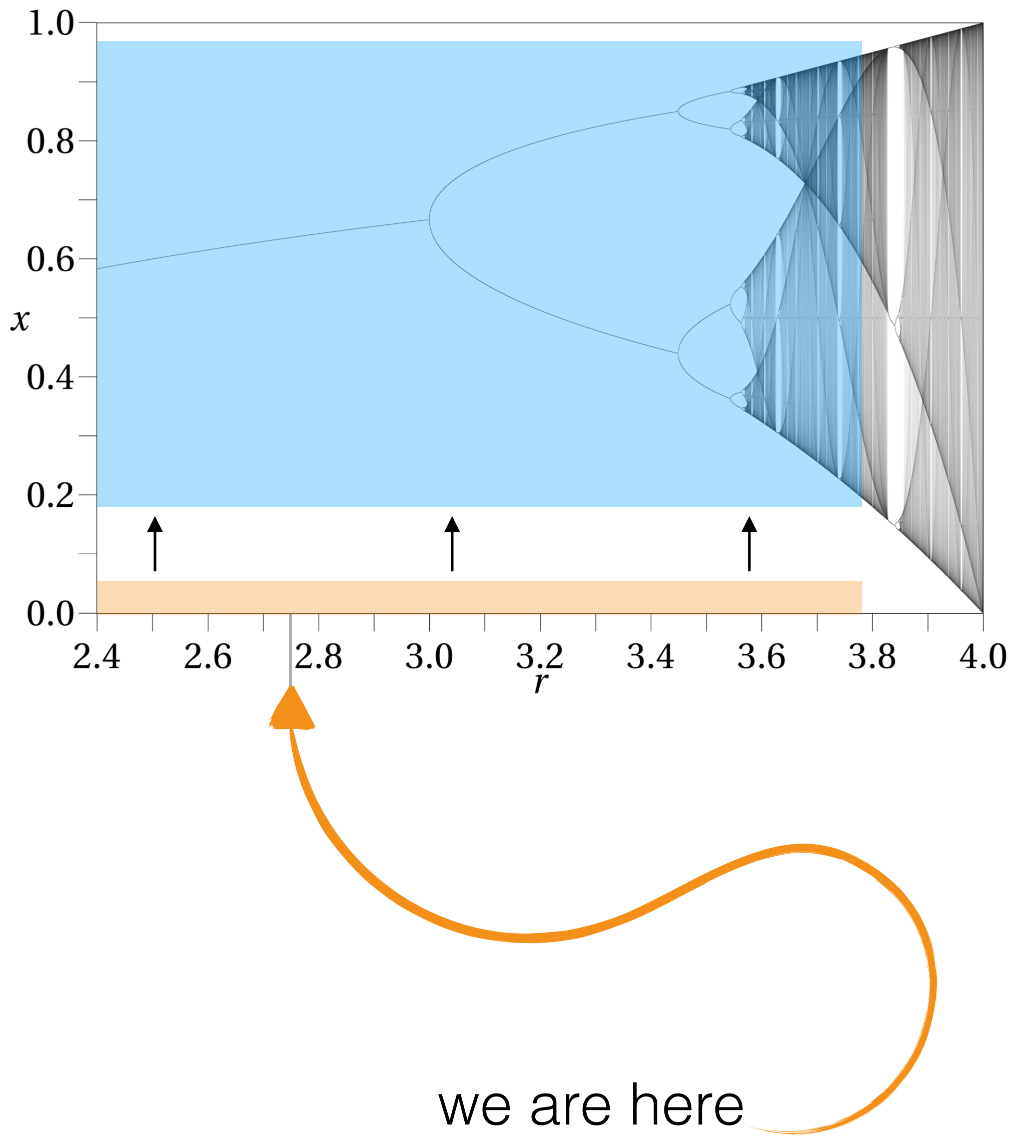
dimension 0 dimension 1 dimension 2

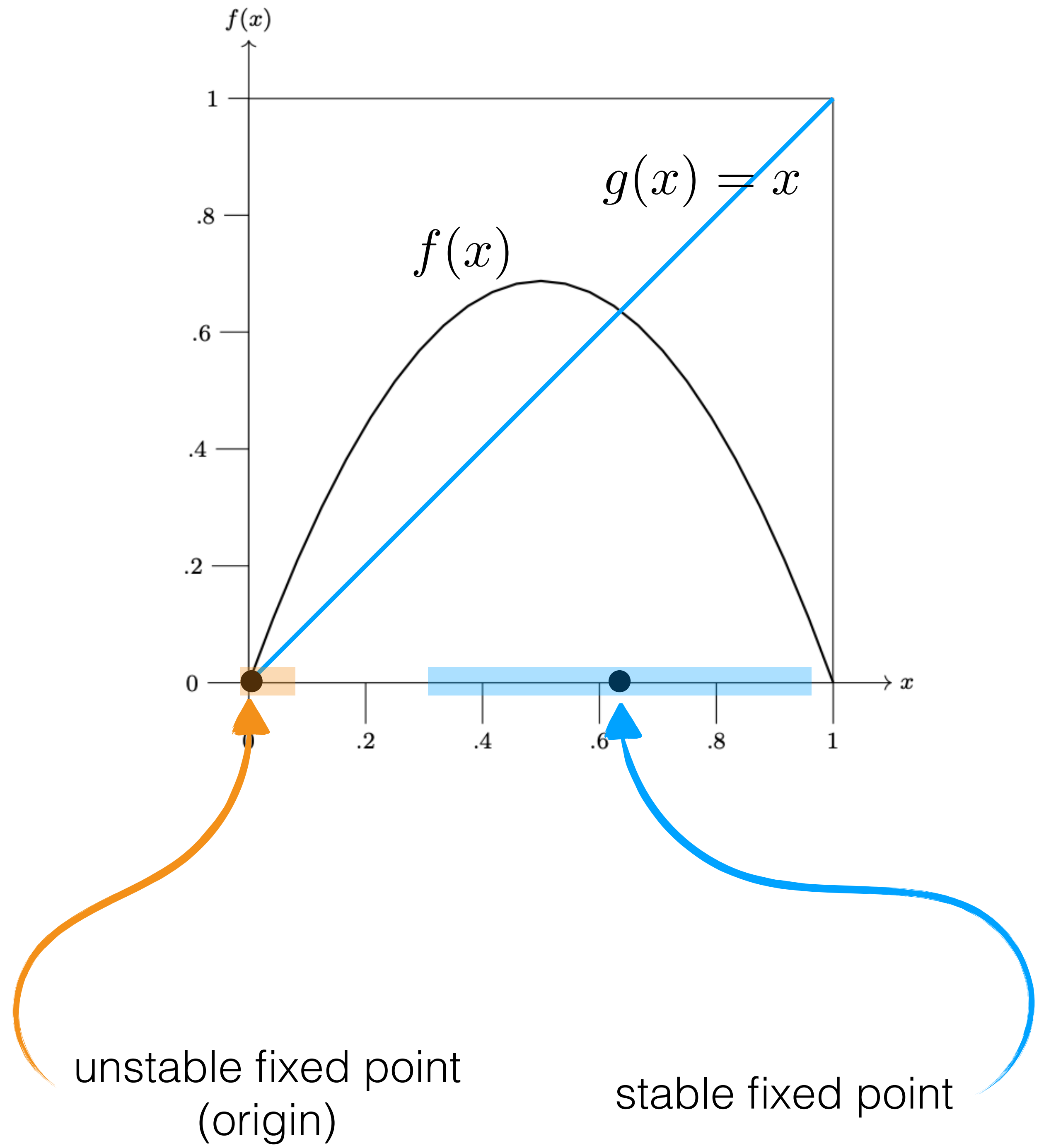
roughly: one-dimensional unstable dynamics

Logistic redux

$r = 2.75$

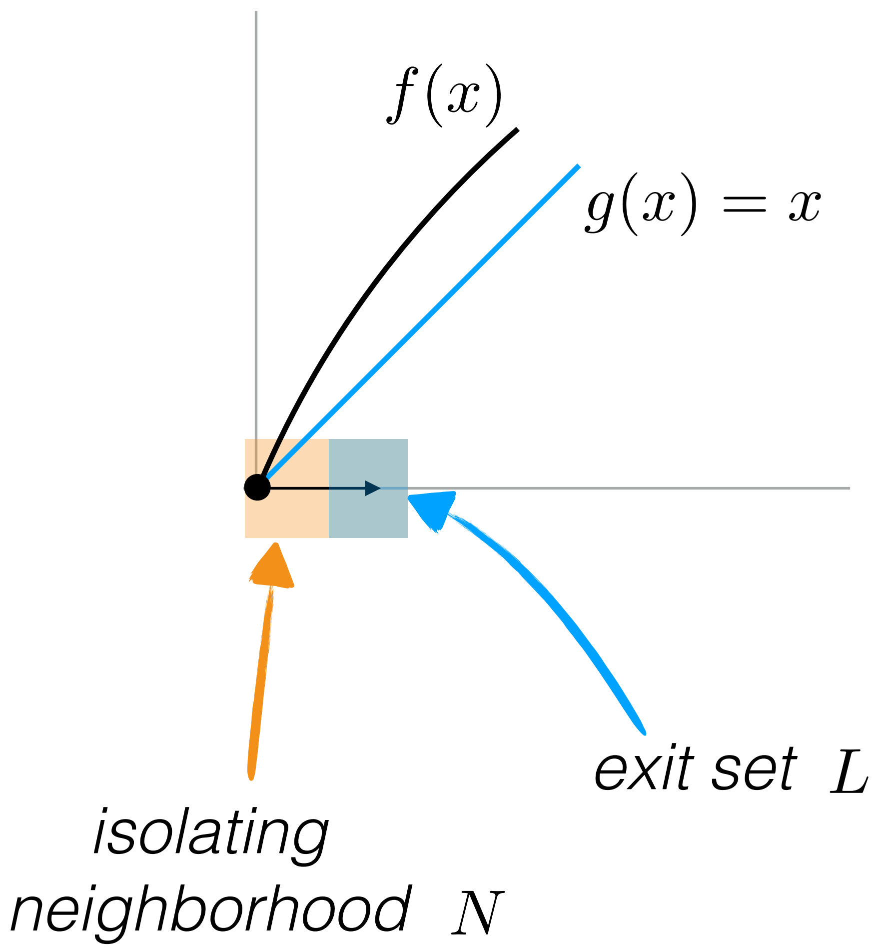
$f(x) = 2.75x(1-x)$





reconstruction via Conley index

local picture near $x = 0$

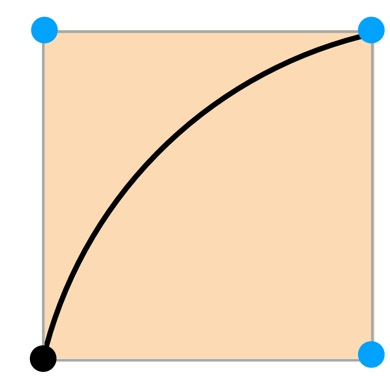


(pointed) quotient space

$(N/L, [L])$

induced self-map

$f_{N,L} \colon (N/L, [L]) \to (N/L, [L])$

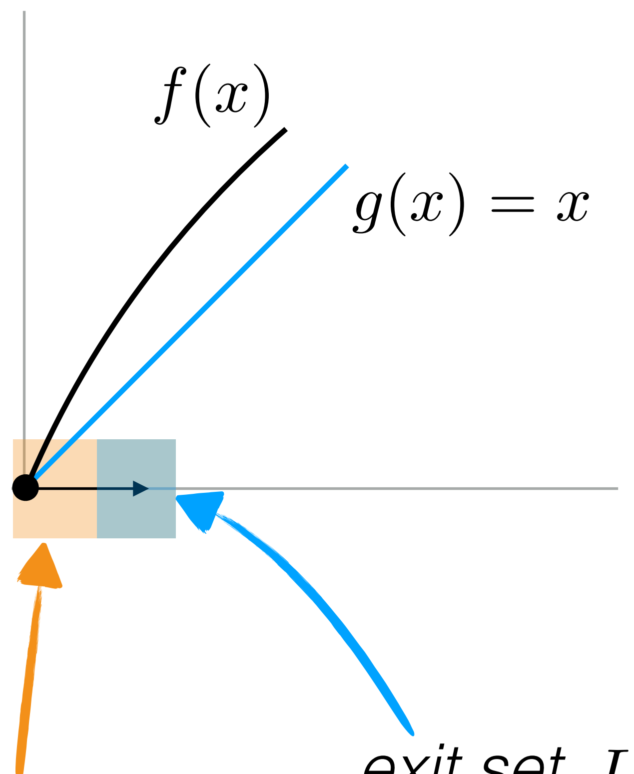


Conley index is the associated linear map

$(f_{N,L})_\bullet \colon H_\bullet(N/L, [L]) \to H_\bullet(N/L, [L])$

reconstruction via Conley index

local picture near $x = 0$



exit set L

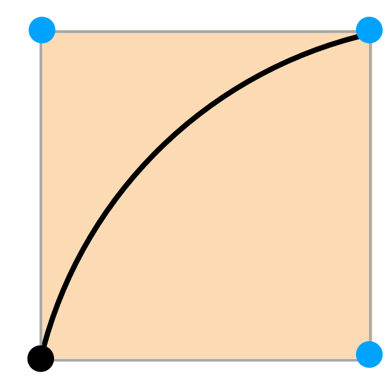
isolating neighborhood N

(pointed) quotient space

$(N/L, [L])$

induced self-map

$f_{N,L} \colon (N/L, [L]) \to (N/L, [L])$



Conley index is the associated linear map

$$(f_{N,L})_\bullet \colon H_\bullet(N/L, [L]) \to H_\bullet(N/L, [L])$$

doing the computation...

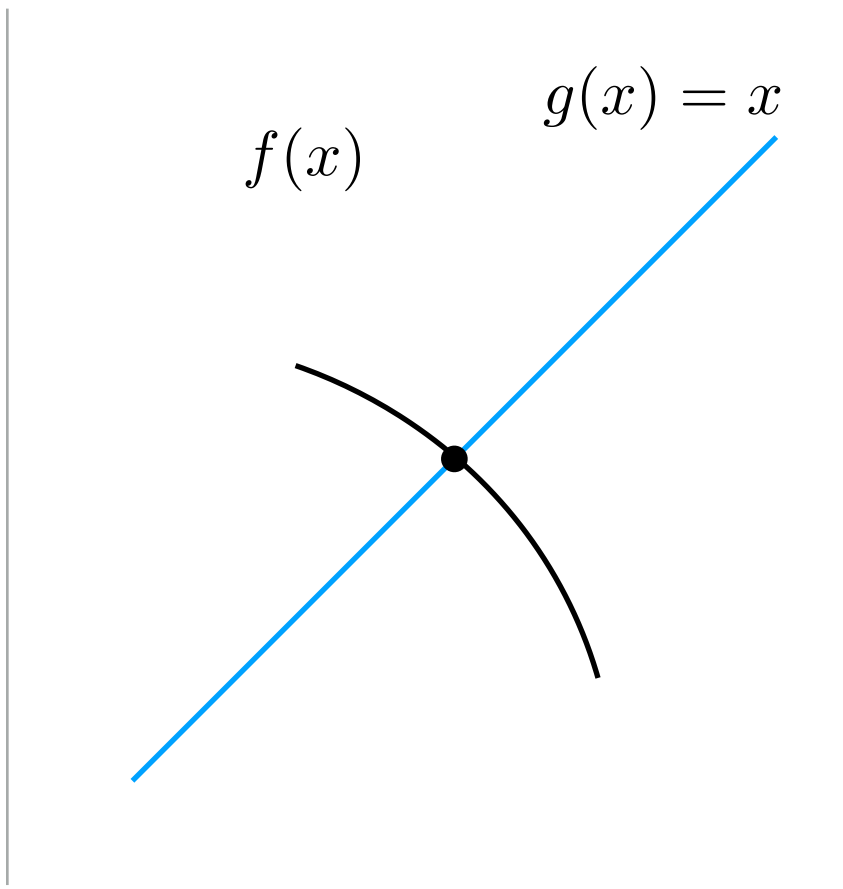
$$H_\bullet(N/L, [L]) = (H_0(N/L, [L]), H_1(N/L, [L]))$$
$$= (0, 0)$$
$$(f_{N,L})_\bullet = (0 \to 0, 0 \to 0)$$

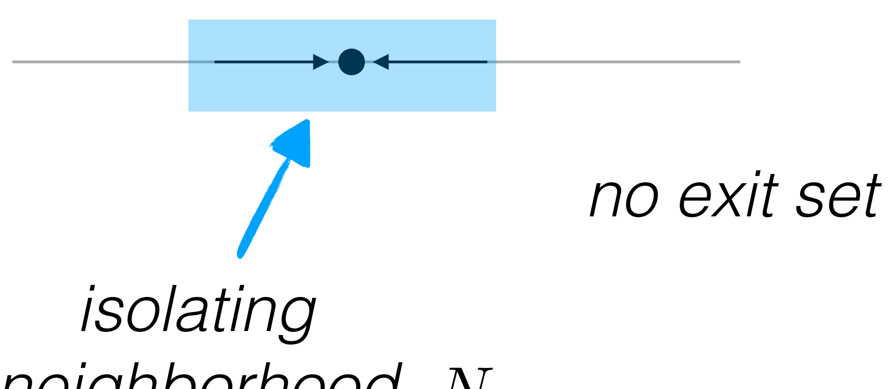
we write the Conley index as $(0, 0)$

(caveat: shift equivalence class)

reconstruction via Conley index II

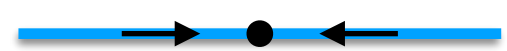
local picture
near attracting fixed point





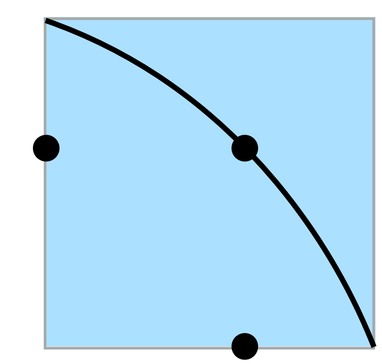
(pointed) quotient space

$(N/L, [L])$



induced self-map

$f_{N,L} \colon (N/L, [L]) \to (N/L, [L])$

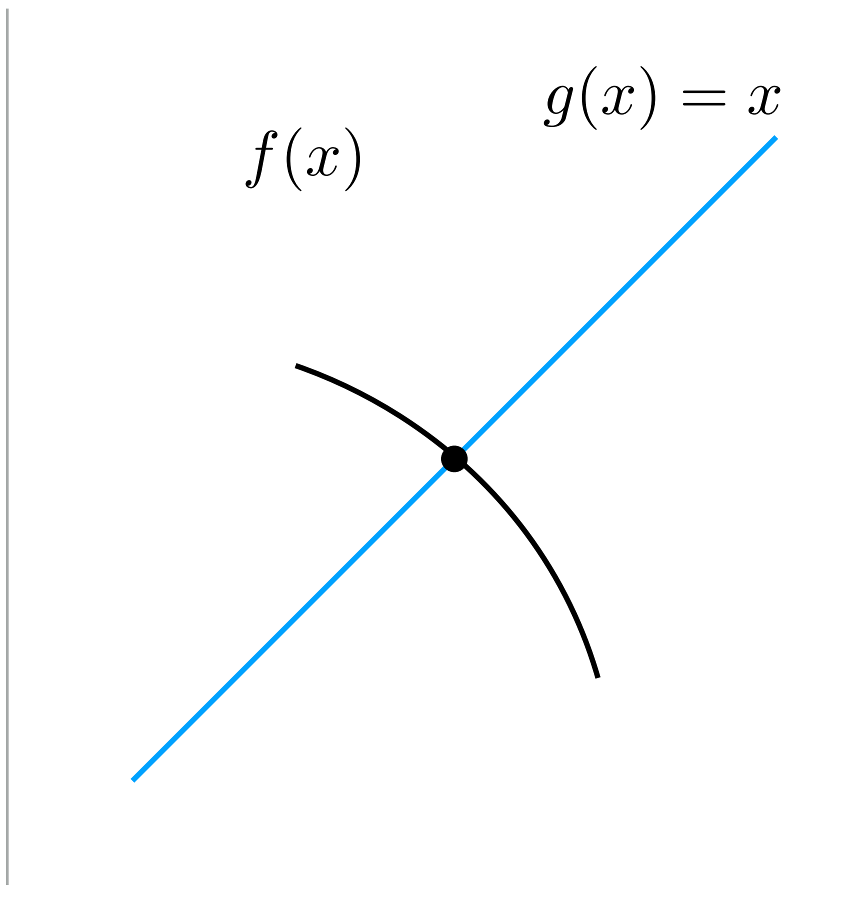


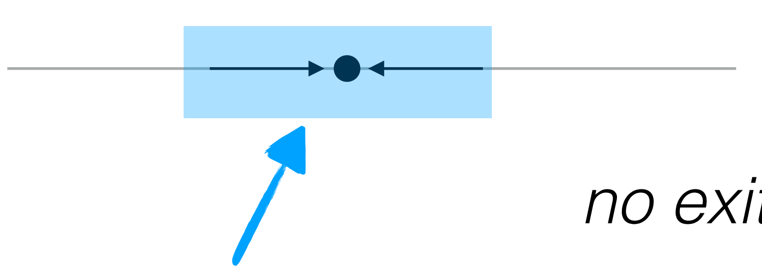
Conley index is the associated linear map

$$(f_{N,L})_\bullet \colon H_\bullet(N/L, [L]) \to H_\bullet(N/L, [L])$$

reconstruction via Conley index II

local picture
near attracting fixed point



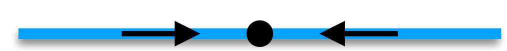


no exit set

isolating
neighborhood N

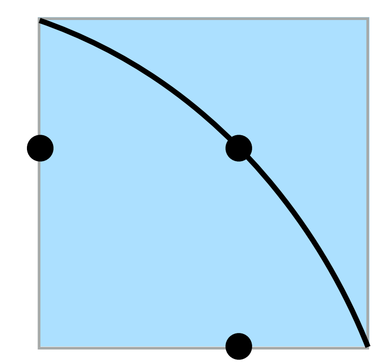
(pointed) quotient space

$(N/L, [L])$



induced self-map

$f_{N,L} \colon (N/L, [L]) \to (N/L, [L])$



Conley index is the associated linear map

$$(f_{N,L})_\bullet \colon H_\bullet(N/L, [L]) \to H_\bullet(N/L, [L])$$

doing the computation…

$$H_\bullet(N/L, [L]) = (\mathbb{F}, 0)$$

$$(f_{N,L})_\bullet = (\mathbb{F} \xrightarrow{\mathrm{id}} \mathbb{F}, 0 \to 0)$$

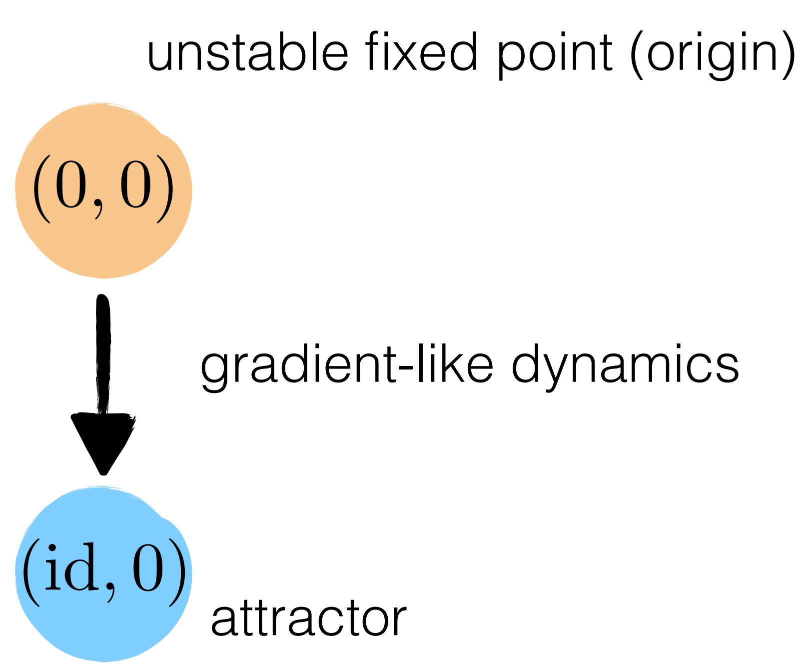
we write the Conley index as $(\mathrm{id}, 0)$

characterize recurrent dynamics over a set of parameters

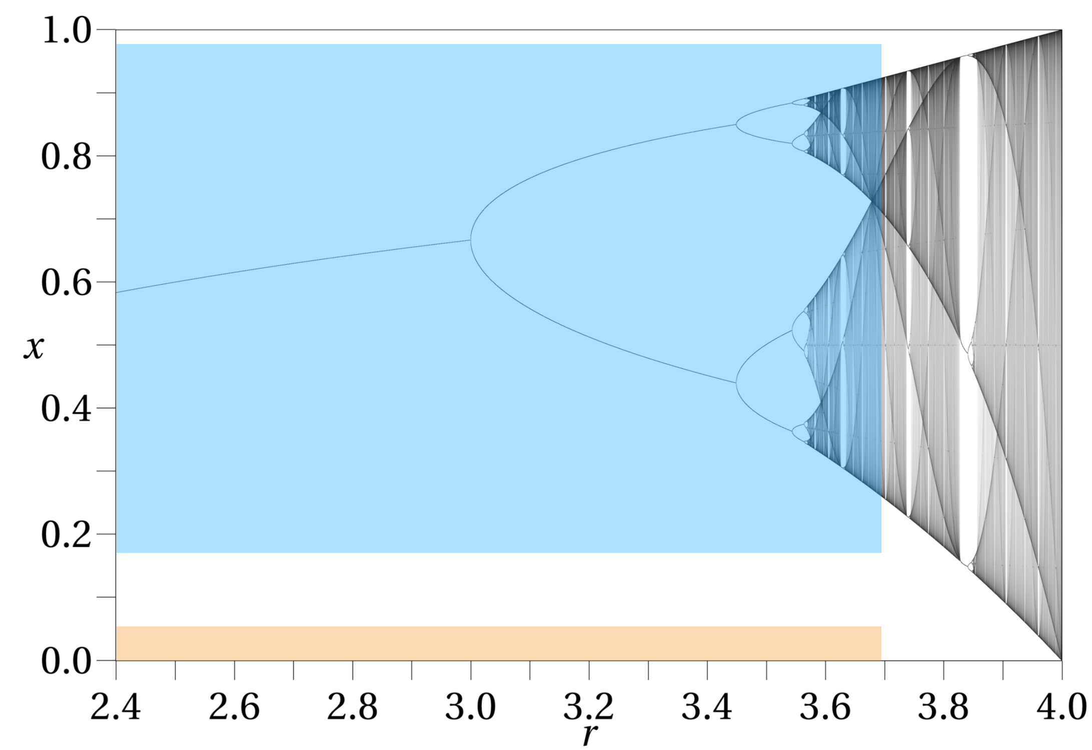
Theorem: If ***N*** is an isolating neighborhood for each ***r*** in a path connected set, then the Conley index associated to ***N*** is the same for all f_r

i.e., robust with respect to parameters

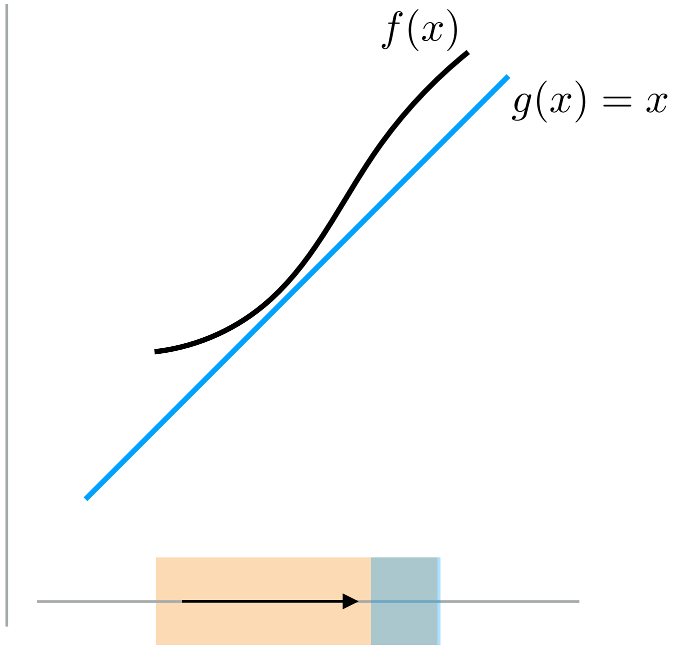
Conley-Morse graph = Morse graph + Conley indices



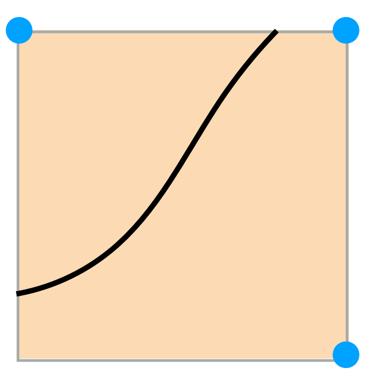
Conley-Morse graph



a zoo of Conley indices

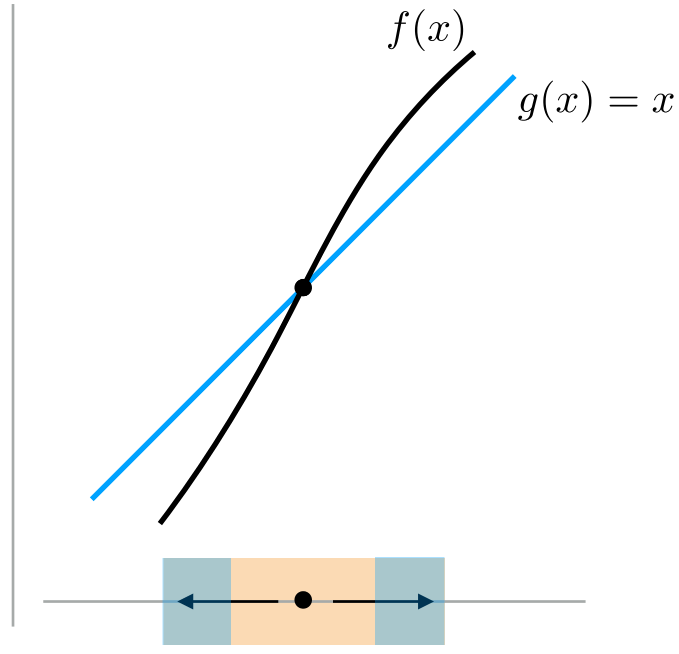


empty invariant set

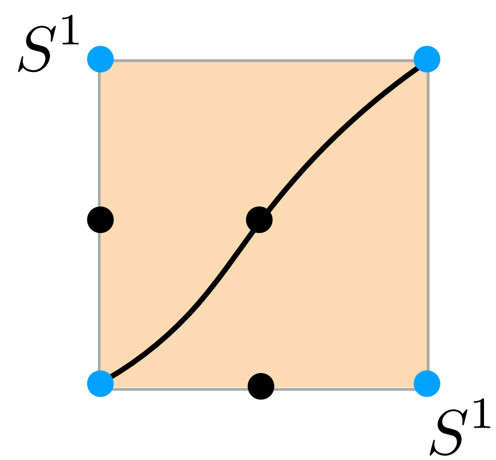


$$H_\bullet(N/L, [L]) = (0, 0)$$

Conley index: $(0, 0)$

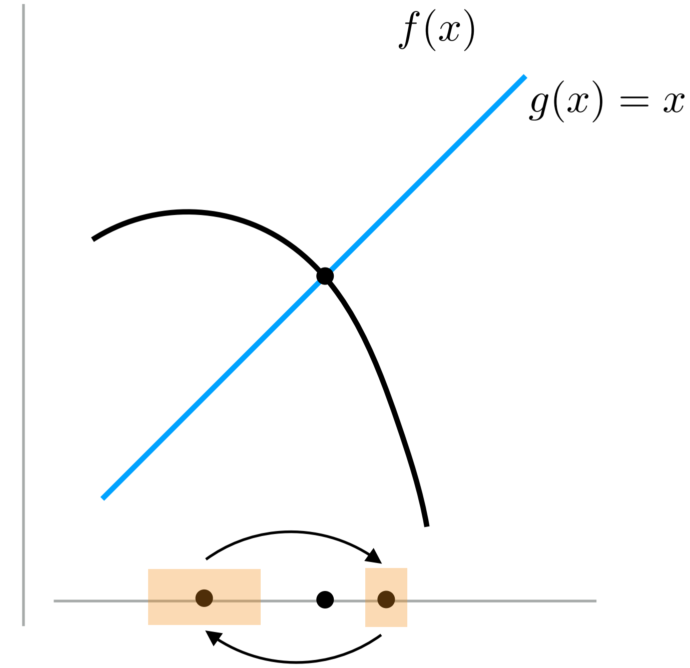


unstable fixed point

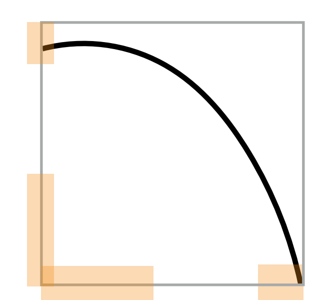


$$H_\bullet(N/L, [L]) = (0, \mathbb{F})$$

$(0, \mathrm{id})$



stable period 2 orbit



$$H_\bullet(N/L, [L]) = (\mathbb{F} \oplus \mathbb{F}, 0)$$

$$\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0 \right)$$

interlude

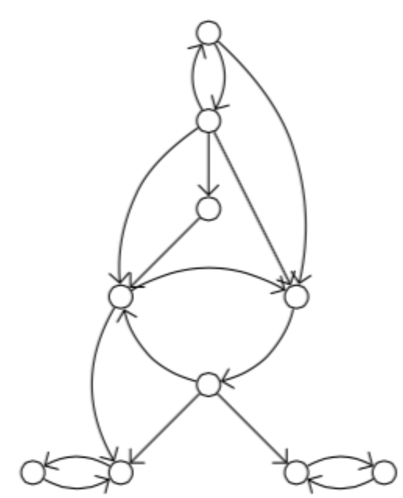
combinatorial dynamics

combinatorial dynamics

vertices = states of the system

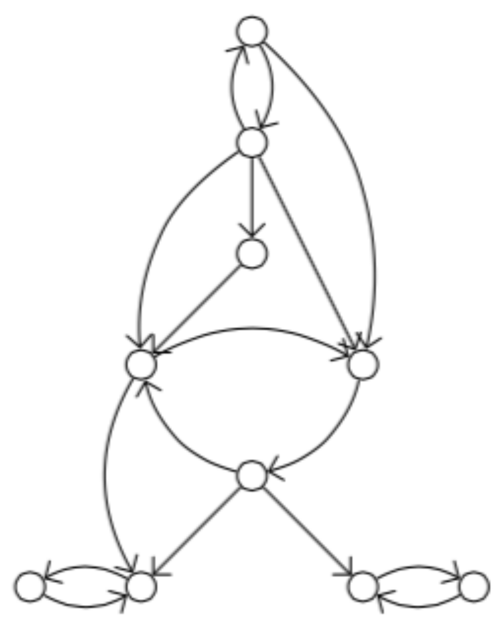
directed edges = possible dynamics

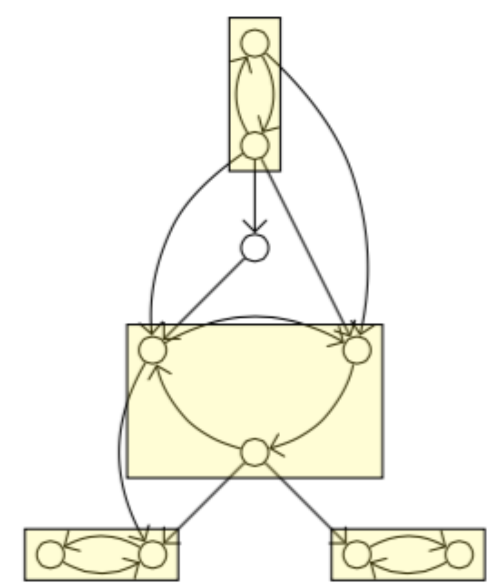
graphs can be manipulated by a computer



decomposition

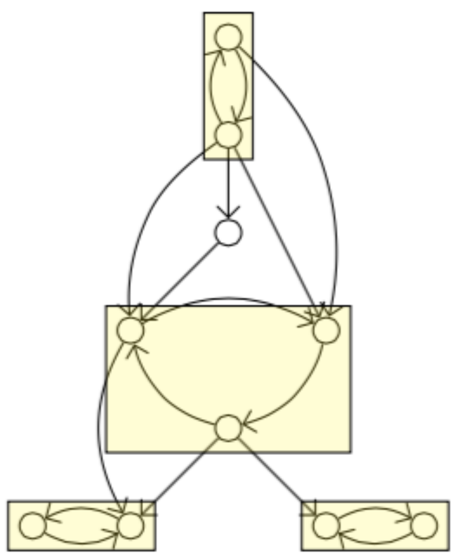
recurrent vs. non-recurrent

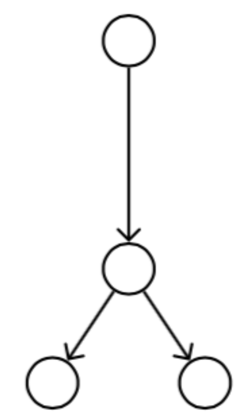




cyclic strongly connected component:
maximal sets of vertices in which any two vertices are connected with a directed path, and contains at least one cycle

decomposition II





(combinatorial) Morse graph

cyclic
strongly connected path components

poset of cyclic
strongly connected components

combinatorialization

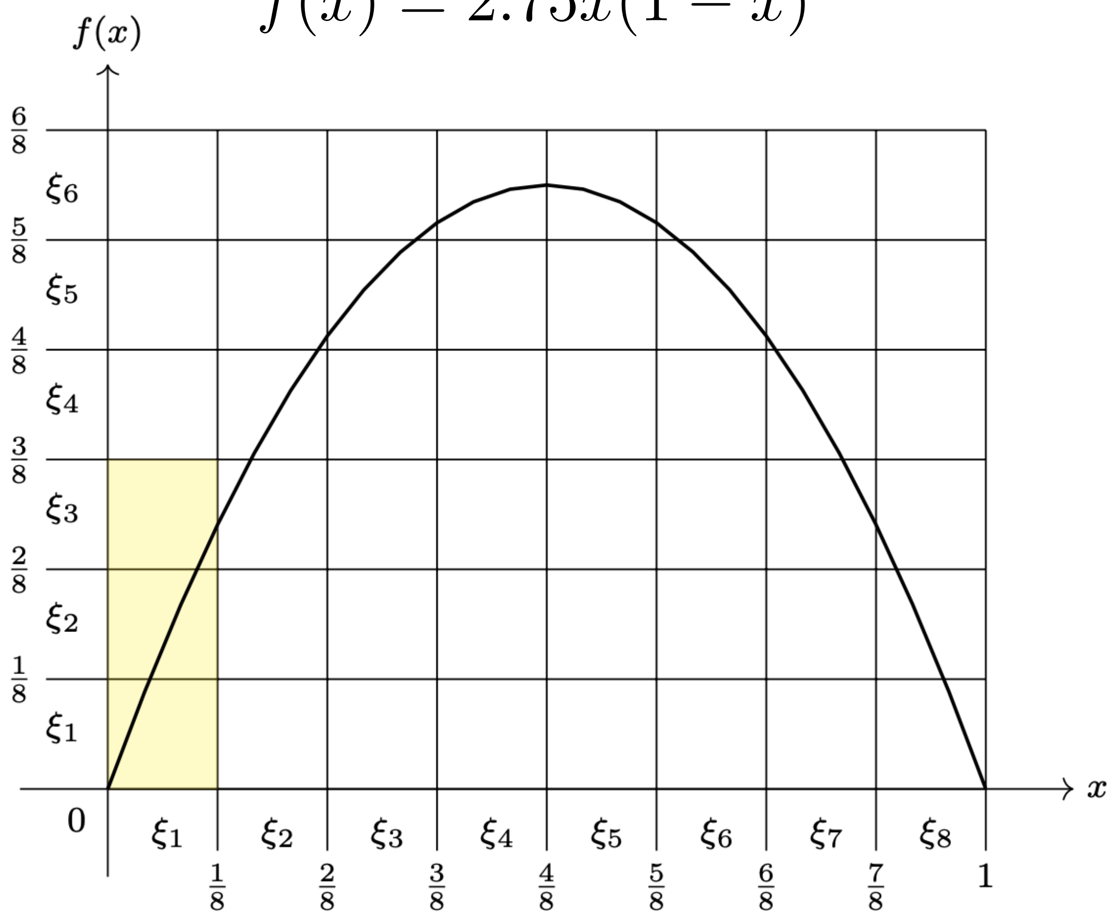
how to go from wiggles to bits and back again

‘Our physical world is wiggly… but in nature wiggles don’t come ‘pre-bitted’. If you want to eat a chicken you have to cut it up — it doesn’t come bitten.’
A. Watts

step 1: discretize phase space

(grid on phase space)

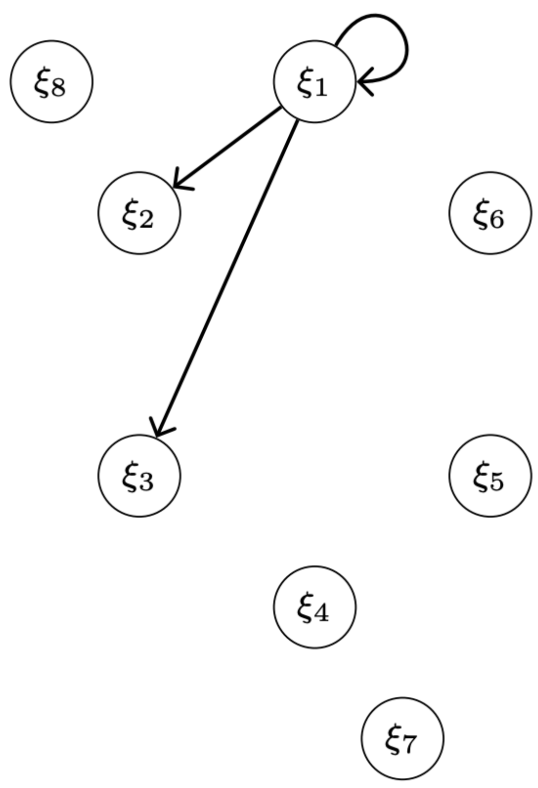
$$f(x) = 2.75x(1-x)$$



step 2: discretize dynamics

directed graph

grid elements are vertices



edge set obeys the property:

$$f(\xi_i) \cap \xi_j \neq \emptyset \implies \xi_i \to \xi_j$$

such a graph is called an outer approximation of f

step 1: discretize phase space

(grid on phase space)

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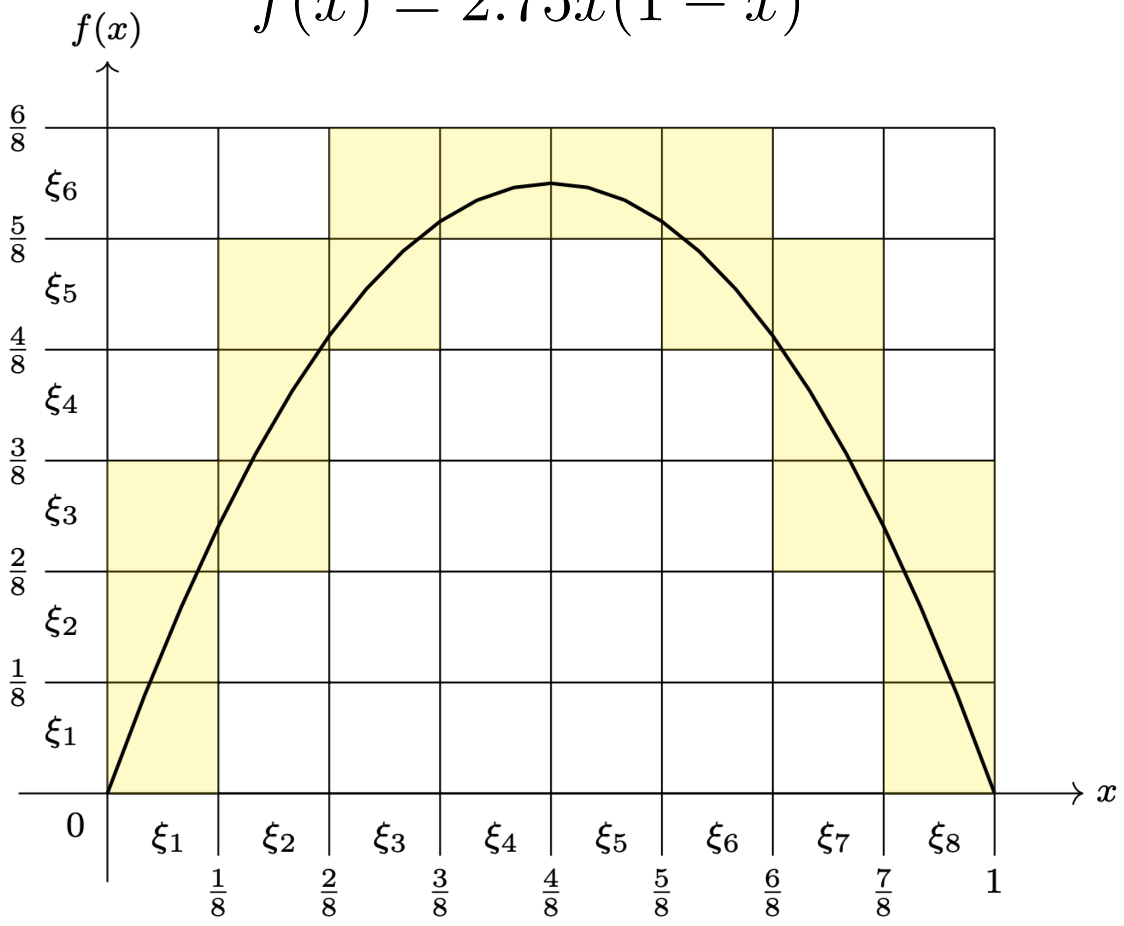

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directed graph

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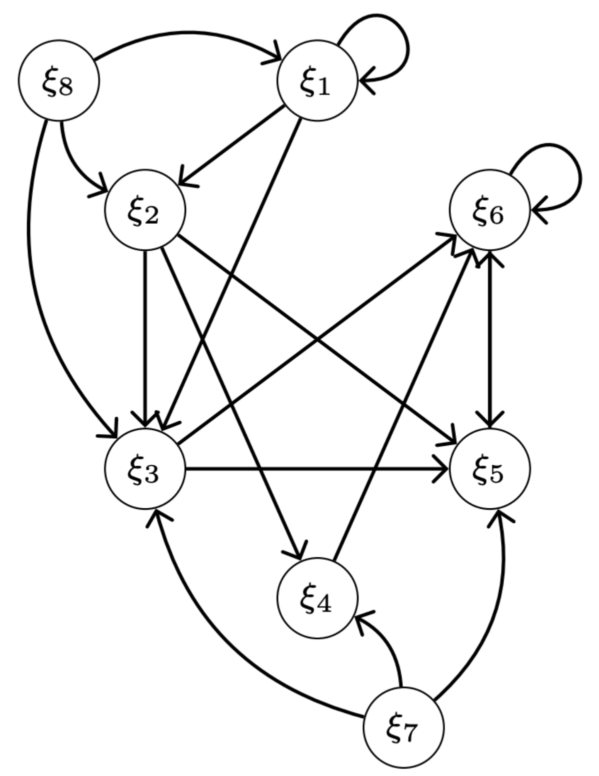

edge set obeys the property:

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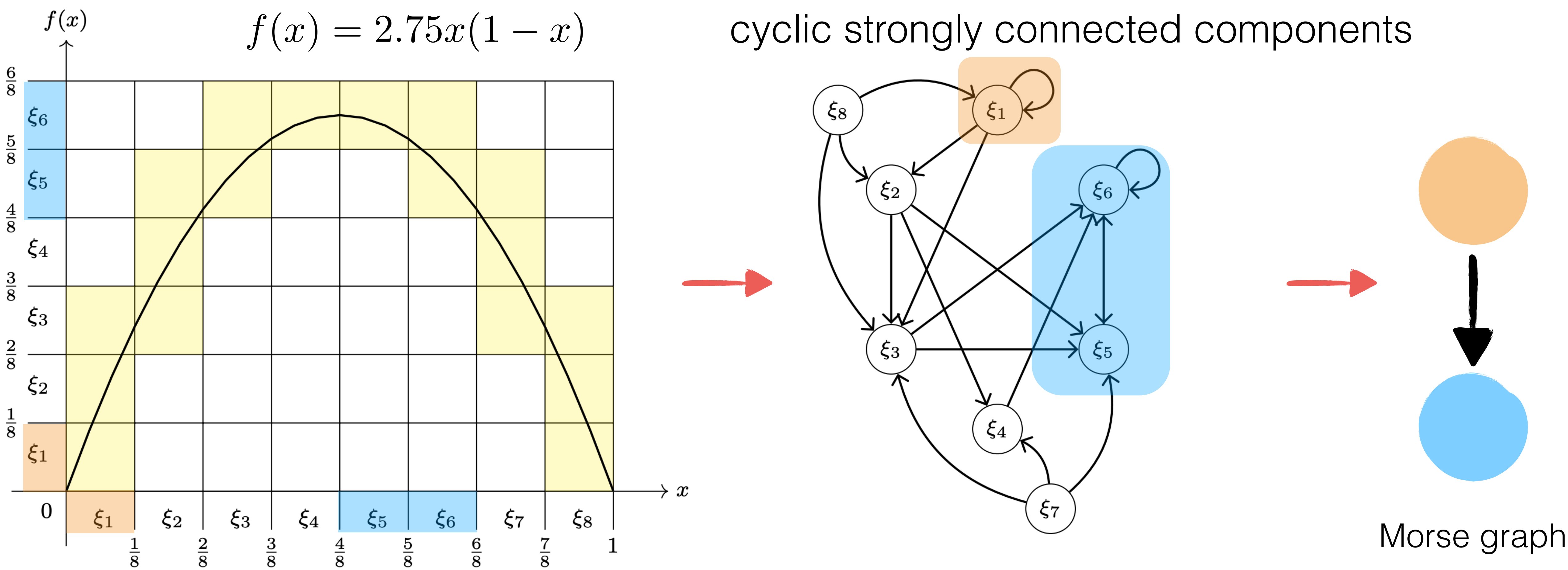
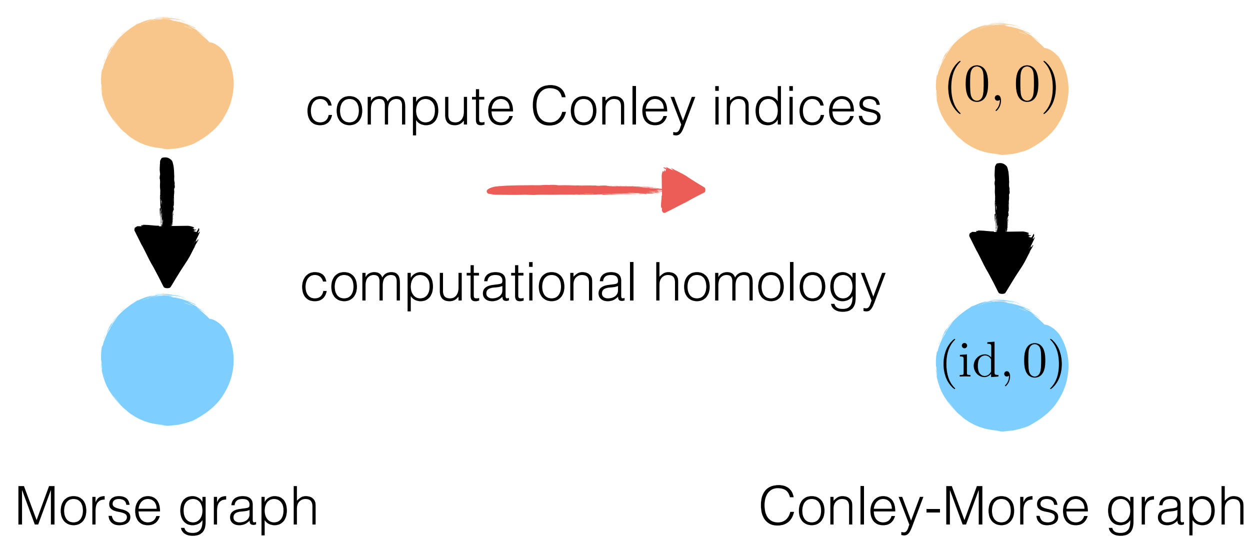
computational recipe





Theorem: Conley-Morse graphs arising from outer approximations are valid for the underlying system f

application…

the global dynamics of games

is there more to a game than its Nash equilibria?
a *long* evening with a friend and two pennies,
how complex is learning to play a game?

‘matching pennies’-style game:
two players (***A*** and ***B***) simultaneously put down a penny, either heads or tails up;
if same face then player ***A*** wins, if opposite faces player ***B*** wins

payoff matrices

$$A = \begin{array}{c c} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \qquad B = \begin{array}{c c} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

suppose ***A*** plays ***H*** and ***B*** plays ***T***, then ***A***’s payoff is $\boldsymbol{A_{HT}}=0$ and ***B***’s is $\boldsymbol{B_{TH}}=1$

player ***A*** strategy: choose ***H*** with probability x_1, choose ***T*** with probability $x_2 = 1 - x_1$
player ***B*** strategy: choose ***H*** with probability y_1, choose ***T*** with probability $y_2 = 1 - y_1$

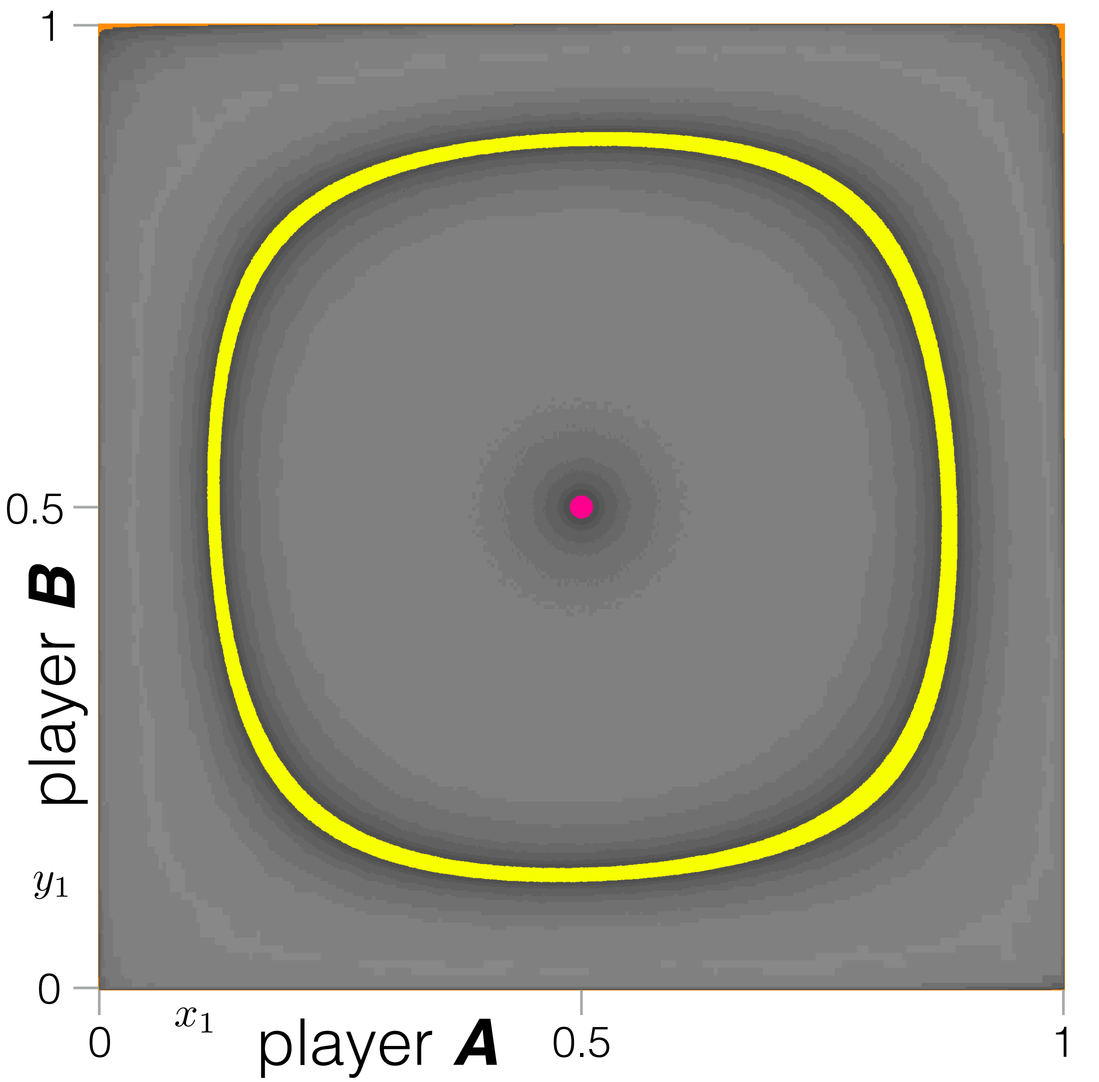
a *learning dynamic* governs how player should update their strategy

$$x_i(t+1) = \frac{x_i(t)^{1-\alpha} e^{\beta(Ay)_i}}{\sum_j x_j(t)^{1-\alpha} e^{\beta(Ay)_j}}, \qquad y_i(t+1) = \frac{y_i(t)^{1-\alpha} e^{\beta(Bx)_i}}{\sum_j y_j(t)^{1-\alpha} e^{\beta(Bx)_j}}.$$

‘Experience Weighted Attraction’

Camerer, C., et al. (1999). Experience–weighted attraction learning in normal form games. *Econometrica.*

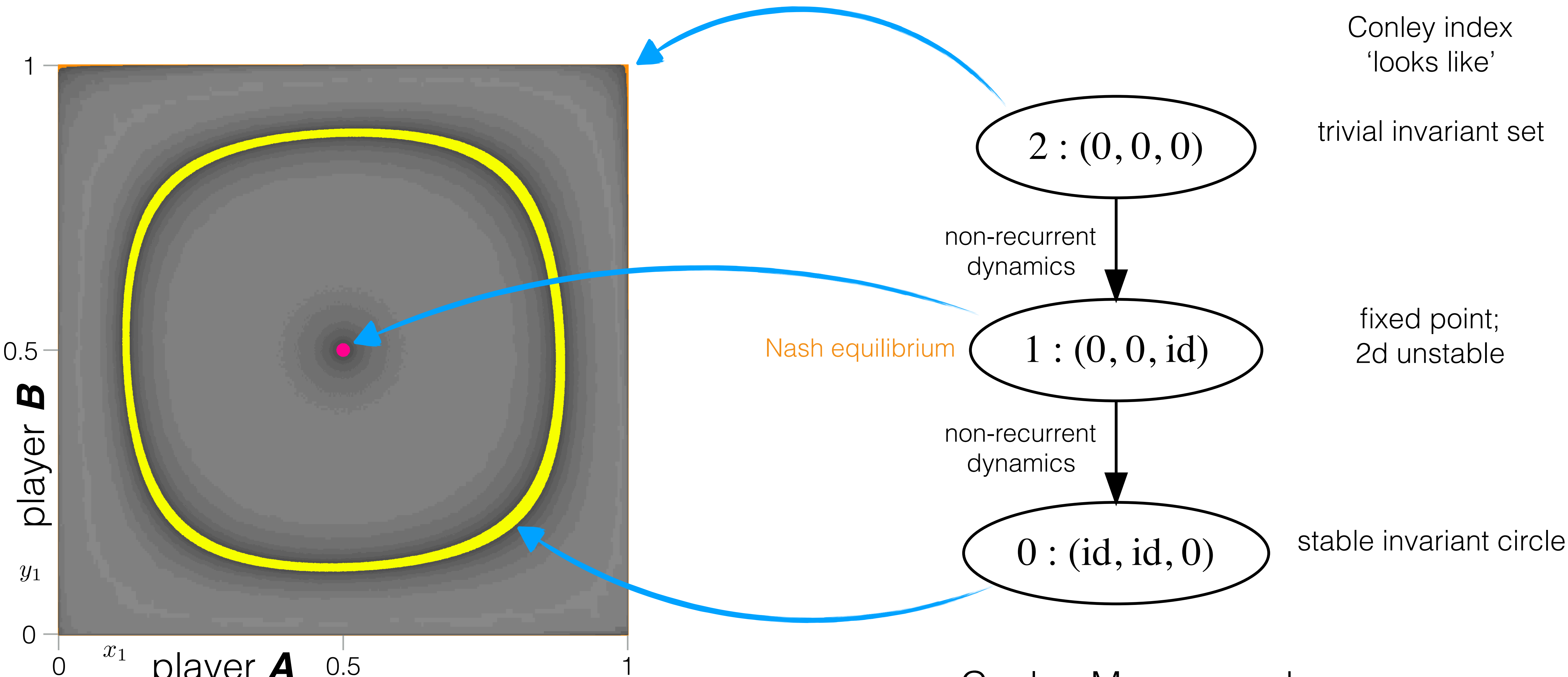
computational recipe for Conley-Morse theory



phase space $X = [0,1] \times [0,1]$

$$\alpha = 0.18, \beta = \sqrt{2}$$

1. discretize phase space at chosen scale
2. build directed graph by *outer approximating* map
3. find cyclic strongly connected components (in color)
4. compute Conley-Morse graph



Conley-Morse graph

phase space $X = [0,1] \times [0,1]$

$$\alpha = 0.18, \beta = \sqrt{2}$$

Given this method of learning the game,
our two players will not converge to the Nash equilibrium!

Instead, they *cycle* around the Nash equilibrium
(groundhog day scenario!).

from zero to one

payoff matrices depend upon parameter: $A = \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix} \quad B = \begin{pmatrix} -r & 1 \\ 1 & -r \end{pmatrix}$

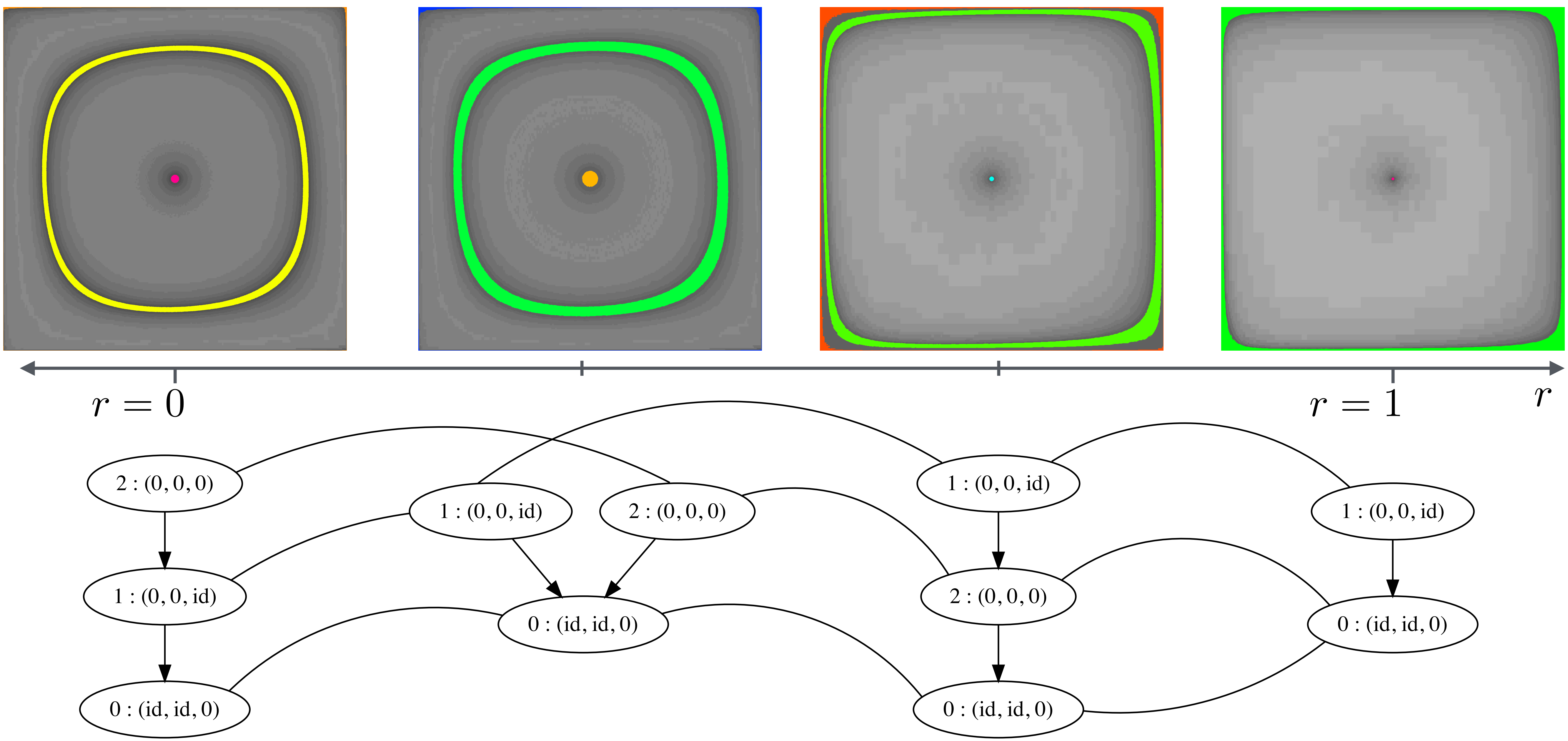
`zero-sum game'

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$A = \begin{pmatrix} 1 & -.1 \\ -.1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -.1 & 1 \\ 1 & -.1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & -.5 \\ -.5 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -.5 & 1 \\ 1 & -.5 \end{pmatrix}$

$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$



returning to our dynamical questions…

too hard

~~for a particular initial condition, what happens in the long run?~~
~~how does the long run behavior depend upon the initial conditions?~~

these may be (partially) answered using Conley theory

is there a fixed point?
is there a periodic orbit?
are there multiple attractors?
how does behavior depend on the parameters within the model?

thank you for your attention

collaborators:
Georgios Piliouras, SUTD

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